**Solution to Problem Set**
**Physics 498TBP**
**by Sinan Arslan**

# 1 Verhulst Equation

### (a)

At the stationary points $a = x_{1s,2s}$, the time derivative of $x$ vanishes, i.e.,

$$x - x^2 = 0, \quad (1)$$
$$x_{1s} = 0, \quad (2)$$
$$x_{2s} = 1. \quad (3)$$

The linear approximation of equation (1, homework) around $x = a = x_{1s,2s}$ is

$$\delta\dot{x} = f(a) + \partial_x f|_a \delta x, \quad (4)$$
$$= (1 - 2a)\delta x. \quad (5)$$

Solution for this equation is

$$\delta x(t) = \delta x(0) e^{(1-2a)t}. \quad (6)$$

and its behavior around the stationary points is

$$\delta x(t) = \begin{cases} \delta x(0)e^{t} & x_{1s} = 0 \\ \delta x(0)e^{-t} & x_{2s} = 1 \end{cases} \quad (7)$$

In the first case $x(t)$ moves away from $x_{1s} = 0$ and in the second case it gets closer to the $x_{2s} = 1$. Accordingly $x_{1s}$ and $x_{2s}$ are stable and unstable stationary points, respectively.

### (b)

The exact solution of the equation

$$\frac{dx}{dt} = x - x^2, \quad (8)$$

can be derived by writing

$$\frac{dx}{x - x^2} = dt, \quad (9)$$

and integrating both sides

$$\int_{x(0)}^{x(t)} \frac{dx}{x - x^2} = \int_0^t dt. \quad (10)$$

From this follows

$$\begin{aligned}
\int\limits_{x(0)}^{x(t)} dx \left(\frac{1}{x} + \frac{1}{1-x}\right) &= t, & (11)\\
\log x - \log(1-x)|_{x_0}^{x(t)} &= t, & (12)\\
\log\frac{x(t)}{1-x(t)} - \log\frac{x_0}{1-x_0} &= t, & (13)\\
\frac{x(t)}{1-x(t)} &= \frac{x_0}{1-x_0}e^t. & (14)
\end{aligned}$$

One can reorganize the equation (14) to obtain the solution for $x(t)$,

$$x(t) = \frac{x_0}{x_0 + (1-x_0)e^{-t}}. \tag{15}$$

The behavior of the solution around the stationary points can be obtained by expanding it near stationary points. For $x \approx 1$, one can express $\delta x(t) = x(t) - 1$ from the equation (15),

$$\delta x(t) = \frac{(x_0 - 1)e^{-t}}{x_0 + (1-x_0)e^{-t}}. \tag{16}$$

For small values of $\delta x(0) = x_0 - 1$ and $\delta x(t) = x(t) - 1$, equation (16),

$$\delta x(t) = \frac{\delta x(0)e^{-t}}{1 + \delta x(0) - \delta x(0)e^{-t}}, \tag{17}$$

can be expanded as follows

$$\delta x(t) \approx \delta x(0)e^{-t}\left(1 - \delta x(0) + \delta x(0)e^{-t}\right) \approx \delta x(0)e^{-t}. \tag{18}$$

Similarly one can derive for $x(t) \approx 0$ and $\delta x(0) = x_0 - 0$,

$$\delta x(t) \approx \delta x(0)e^{t}. \tag{19}$$

Equations (18) and (19) are identical to those derived in section (a).

## (c)

We can rewrite equation (3, homework) as

$$x_{n+1} = \frac{x_n}{(1-h) + hx_n}. \tag{20}$$

From this follows

$$x_{n+1}^{-1} - 1 = (1-h)x_n^{-1} + h - 1, \tag{21}$$

and

$$x_{n+1}^{-1} - 1 = (1-h)(x_n^{-1} - 1). \tag{22}$$

We can now define a new variable, $u_n = x_n^{-1} - 1$, so that the recursion relation can be simplified as follows:

$$u_n = (1-h)u_{n-1}. \tag{23}$$

The solution of this equation is

$$u_n = (1-h)^n u_0. \tag{24}$$

The answer we seek is obtained by switching back to $x_n$,

$$x_n^{-1} = 1 - (1 - x_0^{-1})(1-h)^n. \tag{25}$$

From this equation one can see that $x_n$ coverges to 1, i.e., $\lim_{n\to\infty} x_n = 1$ for $1 \geq h > 0$. $x_n$ still tends to 1 for $2 > h \geq 1$, but it oscillates around 1, so probably $h \geq 1$ is not a good approximation although it still converges to 1.

Replacing n with $t/h$ and $x_n$ with $x(t)$ in the equation (25) yields

$$x(t) = \frac{1}{1 - (1 - x_0^{-1})(1-h)^{t/h}}. \tag{26}$$

The term $(1-h)^{t/h}$ becomes $e^{-t}$ in the limit where $h \to 0$. One can prove this in the following way:

$$\lim_{h\to 0}(1-h)^{t/h} = \exp\left[\log\left(\lim_{h\to 0}(1-h)^{t/h}\right)\right], \tag{27}$$

$$= \exp\left[\lim_{h\to 0}\log(1-h)^{t/h}\right], \tag{28}$$

$$= \exp\left[\lim_{h\to 0}\frac{\log(1-h)}{\frac{h}{t}}\right]. \tag{29}$$

One can use l'Hospital's Rule, since both the numerator and the denominator in the equation (29) is zero when $h = 0$.

$$\lim_{h\to 0}(1-h)^{t/h} = \exp\left[\lim_{h\to 0}\frac{\frac{\partial}{\partial_h}\log(1-h)}{\frac{\partial}{\partial_h}\frac{h}{t}}\right], \tag{30}$$

$$= \exp\left[\lim_{h\to 0}\frac{\frac{-1}{(1-h)}}{\frac{1}{t}}\right], \tag{31}$$

$$= \exp\left[-t\right]. \tag{32}$$

Hence the equation (26) becomes

$$x(t) = \frac{x_0}{x_0 + (1-x_0)e^{-t}}. \tag{33}$$

in the limit where $h \to 0$. Equation (33) is same as the exact solution we obtained in section (b).

## 2 Limit Cycle

Equation (5, homework) can be expressed in polar coordinates by using the substitutions

$$x = r\cos\theta, \tag{34}$$

$$y = r\sin\theta, \tag{35}$$

$$\dot{x} = \dot{r}\cos\theta - r\sin\theta\,\dot{\theta}, \tag{36}$$

$$\dot{y} = \dot{r}\sin\theta + r\cos\theta\,\dot{\theta}. \tag{37}$$

Thus the equation (5, homework) in polar coordinates becomes

$$\dot{r}\cos\theta - r\sin\theta\,\dot{\theta} = r\sin\theta + r\cos\theta f(r), \tag{38}$$

$$\dot{r}\sin\theta + r\cos\theta\,\dot{\theta} = -r\cos\theta + r\sin\theta f(r). \tag{39}$$

By multiplying both sides of the equations (38) and (39) by $\cos\theta$ and $\sin\theta$, respectively, then adding them, one obtains

$$\dot{r} = rf(r). \tag{40}$$

Similarly one can derive

$$\dot{\theta} = -1. \tag{41}$$

Equation (41) means that the system $\{x(t), y(t)\}$ rotates around origin with a constant angular velocity $-1$. We can now find the solutions of equation (40) for three cases of $f(r)$.

For $f(r) = 1 - r^2$, equation (40) becomes

$$\frac{dr}{dt} = r(1 - r^2). \tag{42}$$

One can write this equation as

$$dr\left(\frac{1}{r} + \frac{1}{2}\left(\frac{1}{1-r} - \frac{1}{1+r}\right)\right) = dt. \tag{43}$$

Carrying out the integral on both sides

$$\int_{r_0}^{r(t)} dr\left(\frac{1}{r} + \frac{1}{2}\left(\frac{1}{1-r} - \frac{1}{1+r}\right)\right) = \int_0^t dt', \tag{44}$$

yields

$$\log r + \frac{1}{2}\left(-\log(1-r) - \log(1+r)\right)\Big|_{r_0}^{r(t)} = t, \tag{45}$$

$$\log r - \frac{1}{2}\log(1-r^2)\Big|_{r_0}^{r(t)} = t, \tag{46}$$

$$\log \frac{r}{\sqrt{(1-r^2)}}\Big|_{r_0}^{r(t)} = t, \tag{47}$$

$$\log \frac{r(t)}{\sqrt{1-r^2(t)}} - \log \frac{r_0}{\sqrt{1-r_0^2}} = t, \tag{48}$$

$$\frac{r(t)}{\sqrt{1-r^2(t)}} = \frac{r_0}{\sqrt{1-r_0^2}}e^t. \tag{49}$$

This expression can be reorganized to get

$$r(t) = \frac{r_0}{\sqrt{r_0^2 + (1-r_0^2)e^{-2t}}}. \tag{50}$$

From equation (50), one can see that $r(t)$ converges to 1 regardless of the initial condition, except $r_0 = 0$. Eventually the system begins to move on a circle of radius 1. On the other hand $r = 0$ is an unstable stationary point: if the initial condition is $r_0 = 0$, $r(t)$ does not change in time, but for $r_0 > 1$, it moves to 1.

For $f(r) = r^2 - 1$, we get the same solution as above if we replace $t$ with $-t$. Therefore the solution is

$$r(t) = \frac{r_0}{\sqrt{r_0^2 + (1-r_0^2)e^{2t}}}. \tag{51}$$

The behavior of this solution for different initial conditions can be summarized:

$$\lim_{t\to\infty} r(t) = \begin{cases} \infty & r_0 > 1 \\ 1 & r_0 = 1 \\ 0 & r_0 < 1 \end{cases} \tag{52}$$

This tells us that the points on the circle of radius $r = 1$ and whose center at the origin are unstable stationary points whereas the origin, $r = 0$ is a stable stationary point.

In case of $f(r) = (1-r^2)^2$ the equation (40) reads

$$\frac{dr}{dt} = r(1-r^2)^2. \tag{53}$$

Gathering each variable on the both sides separately and integrating yields

$$\int_{r_0}^{r(t)} \frac{dr^2}{2r^2(1-r^2)^2} = \int_0^t dt'. \tag{54}$$

By using the substitutions

$$r^2 = \frac{R}{(R-1)}, \tag{55}$$

$$dr^2 = -\frac{dR}{(1-R)^2}, \tag{56}$$

one can get

$$\int_{R_0}^{R(t)} \left(\frac{1}{R} - 1\right) dR = 2t, \tag{57}$$

$$\log R - R|_{R_0}^{R(t)} = 2t, \tag{58}$$

$$\log \left.\frac{R}{e^R}\right|_{R_0}^{R(t)} = 2t, \tag{59}$$

$$\frac{R(t)}{e^{R(t)}} = \frac{R_0}{e^{R_0}} e^{2t}. \tag{60}$$

Let us call this function $G(r)$, then the solution is

$$G(R(t)) = \frac{R(t)}{e^{R(t)}} = \frac{R_0}{e^{R_0}} e^{2t} \tag{61}$$

where $R(t) = r(t)^2/(r^2(t) - 1)$. From equation (61), one can see that $G(R)$ has always the tendency to increase in time for positive initial values $G(R_0) > 0$ or to decrease for negative values of $G(R_0)$.

From equation (53), we know that the stationary points are $r = 0$ and $r = 1$. If $r_0 = 0$, then $R_0 = G(R_0) = 0$ and the system does not move from $r = 0$. Where $1 > r_0 > 0$, $G(R_0)$ is negative and the system moves towards $r = 1$, because $\lim_{r(t)\to 1^-} G(R(t)) = -\infty$. So $r = 0$ is an unstable stationary point whereas $r = 1^-$ is a stable stationary point ($1^-$ corresponds to a value that approaches 1 from left). The behavior of the system is depicted in figure 1, based on the function $G(r)$

If the system is at a point such that $r > 1$, which means $G(R) > 0$ then the system goes away from the point $r = 1$ , because $r$ has to increase in order for $G(R)$ to increase by time, which is the condition imposed by equation (61). So the $r = 1^+$ is an unstable stationary point.

# 3 Bonhoefer-van der Pol Equation

## (a)

Since $\dot{x}_1(t)$ and $\dot{x}_2(t)$ are zero at the stationary points $(x_{1s}, x_{2s})$, equation (6, homework) becomes

$$f_1(x_{1s}, x_{2s}) = c\left(x_{2s} + x_{1s} - \frac{x_1s^3}{3} + z\right) = 0, \tag{62}$$

$$f_2(x_{1s}, x_{2s}) = -\frac{1}{c}\left(x_{1s} + bx_{2s} - a\right) = 0. \tag{63}$$

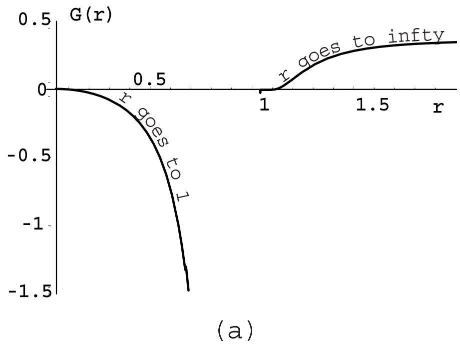


Figure 1: $G(r)$ *vs.* $r$

From these equations we get

$$x_{2s} = \frac{1}{b}(a - x_{1s}) \tag{64}$$

$$0 = \frac{1}{b}(a - x_{1s}) + x_{1s} - \frac{x_{1s}^3}{3} + z \tag{65}$$

which has one real solution $(x_{1s}, x_{2s})$ for a certain value of $z$. It is plotted in figure 2 for values of $z$ between $-2$ and 2.

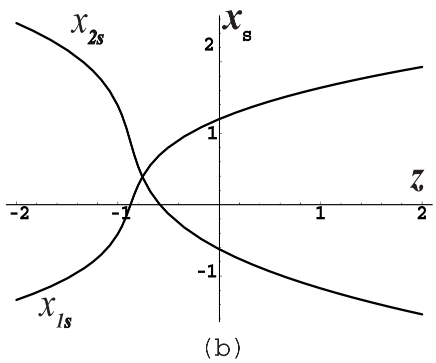


Figure 2: Stationary points $x_{1s}$ and $x_{2s}$ *vs.* $z$

## (b)

Elements of the matrix $\mathbf{M}$ in the equation $(\delta x_1 = x_1 - x_{1s}, \delta x_2 = x_2 - x_{2s})$,

$$\delta \dot{\mathbf{x}} = \mathbf{M} \delta \mathbf{x}, \tag{66}$$

can be calculated by using $M_{jk} = \partial_k f_j(x_1, x_2)|_{\mathbf{x_s}}$,

$$\mathbf{M} = \begin{pmatrix} c(1 - x_{1s}^2) & c \\ -1/c & -b/c \end{pmatrix}. \tag{67}$$

If the eigenvalues and the eigenvectors of the matrix $\mathbf{M}$ are denoted by $\lambda_{1,2}$ and $\mathbf{m_{1,2}}$, respectively, then the solution for the equation (66) is given by

$$\delta\mathbf{x}(t) = c_1 \mathbf{m_1} e^{\lambda_1 t} + c_2 \mathbf{m_2} e^{\lambda_2 t}. \tag{68}$$

where $c_1$ and $c_2$ are determined by the initial condition $\delta\mathbf{x}(0)$. One can easily tell from this equation that around a stable stationary point $(x_{1s}, x_{2s})$, $\lambda_{1,2}$ must have negative real parts.

For $z = 0$, the stationary point is $\mathbf{x}_s = (1.20, -0.62)$. The corresponding matrix and the eigenvalues are given by

$$\mathbf{M} = \begin{pmatrix} -1.32 & 3 \\ -0.33 & -0.27 \end{pmatrix}, \tag{69}$$

$$\lambda_{1,2} = -0.79 \pm 0.85\, i. \tag{70}$$

So the stationary point $\mathbf{x}_s = (1.20, -0.62)$ is stable due to the $e^{-0.79t}$ term in the solution. With similar arguments, for $z = -0.4$, the stationary point $\mathbf{x}_s = (0.91, -0.26)$ is found to be unstable, because the real part of the eigenvalues of the corresponding matrix $\mathbf{M}$,

$$\mathbf{M} = \begin{pmatrix} 0.53 & 3 \\ -0.33 & -0.27 \end{pmatrix}, \tag{71}$$

are positive, i.e.,

$$\lambda_{1,2} = 0.13 \pm 0.92\, i. \tag{72}$$

## (c)

Using the discretized form of the differential equation given in the problem,

$$\mathbf{x}_{n+1} = \mathbf{x}_n + \Delta t\, f(\mathbf{x}_n), \tag{73}$$

one can get curves like in figure 3 for $z = 0$ and $z = -0.4$, with several starting points. Corresponding stationary points for these $z$ values, i.e., $(1.20, -0.62)$ and $(0.91, -0.26)$ are marked in the plots.

## (d)

When we include the noise term,

$$\mathbf{x}_{n+1} = \mathbf{x}_n + \Delta t\, f(\mathbf{x}_n) + \sigma\sqrt{\Delta t} \begin{pmatrix} \zeta_1(n) \\ \zeta_2(n) \end{pmatrix} \tag{74}$$

trajectories will look like in figure 4. In `mathematica` , one way of getting gaussian distributed random numbers, $\zeta_j(n)$, is using the following code:

```
<< Statistics‘ContinuousDistributions‘
ndist = NormalDistribution[0, 1] %% where the mean and the standard deviation
%% of the distribution are specified respectively.
ζ = Random[ndist]
```

$x_1(t)$ for the case $z = -0.4$ is plotted separately in figure 5.

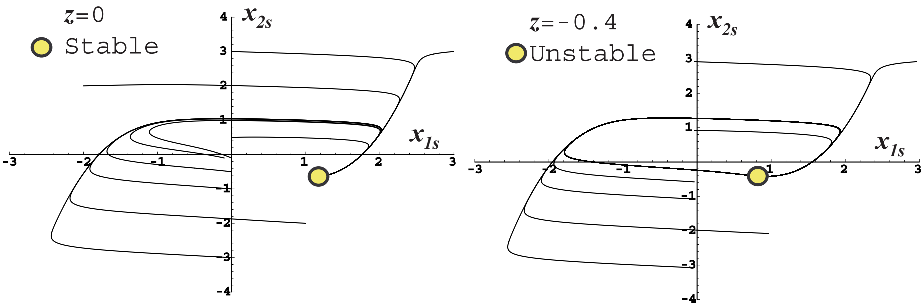


Figure 3: $x_1(t), x_2(t)$ trajectories for different starting points (open ends of the trajectories). In case of a stable stationary point, all the trajectories come to rest at the stationary point whereas in case of an unstable stationary point the system begins a cyclic motion, regardless of its starting point.

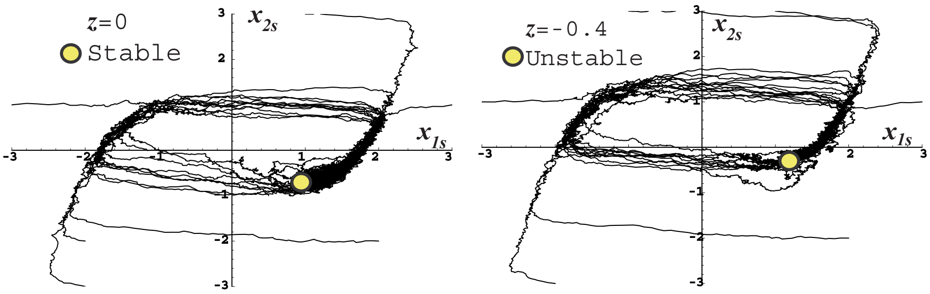


Figure 4: $x_1(t), x_2(t)$ trajectories of the system, in case of gaussian random noise for several initial conditions ($\sigma = 0.15$, $\Delta t = 0.01$ ).

# 4 Cable Equation

Solution for the cable equation

$$v(x,t) = \sqrt{\frac{2}{a}} \sum_{n=1,3,\ldots} \alpha_n(t) \cos \frac{n\pi x}{2a} \tag{75}$$

obeys the boundary conditions, i.e.,

$$\partial_x v(x,t)|_{x=0} = \sqrt{\frac{2}{a}} \sum_{n=1,3,\ldots} \frac{n\pi}{2a} \alpha_n(t) \sin \frac{n\pi x}{2a} \bigg|_{x=0}, \tag{76}$$

$$= 0 \tag{77}$$

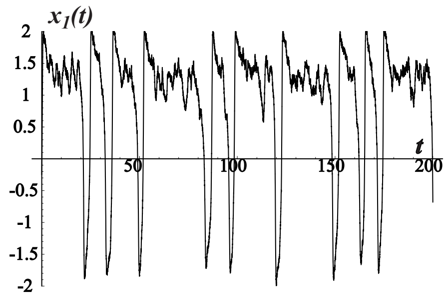


Figure 5: $x_1(t)$ vs. $t$, for the case $z = -0.4$. ($\sigma = 0.15$, $\Delta t = 0.01$).

and

$$v(a,t) = \sqrt{\frac{2}{a}} \sum_{n=1,3,\ldots} \alpha_n(t) \cos\frac{n\pi}{2}, \tag{78}$$

$$= 0. \tag{79}$$

Inserting the general solution (75) into the cable equation provides us with the solution that governs $\alpha_n(t)$:

$$0 = (\partial_t - \partial_x^2 + 1) v(x,t) \tag{80}$$

$$= \sqrt{\frac{2}{a}} \sum_{n=1,3,\ldots} \left( \dot{\alpha}_n(t) \cos\frac{n\pi x}{2a} + \alpha_n(t) \left(\frac{n\pi}{2a}\right)^2 \cos\frac{n\pi x}{2a} + \alpha_n(t) \cos\frac{n\pi x}{2a} \right) \tag{81}$$

$$= \sqrt{\frac{2}{a}} \sum_{n=1,3,\ldots} \left( \dot{\alpha}_n(t) + \left(\frac{n\pi}{2a}\right)^2 \alpha_n(t) + \alpha_n(t) \right) \cos\frac{n\pi x}{2a} \tag{82}$$

In order for above summation to vanish, the term in the parenthesis must vanish. One can prove this by integrating both sides of the equation (82) with $\cos(m\pi x/2a)$ where $m = 1, 3, 5, \ldots$:

$$0 = \sqrt{\frac{2}{a}} \sum_{n=1,3,\ldots} b_n \int_0^a \cos\frac{n\pi x}{2a} \cos\frac{m\pi x}{2a} dx \tag{83}$$

where $b_n$ is the term in the parenthesis in equation (82). The integral in the equation (83) can be calculated by using the identity

$$\cos A \cos B = \frac{1}{2} \left( \cos(A+B) + \cos(A-B) \right).$$

$$\int_0^a dx \cos\frac{n\pi x}{2a} \cos\frac{m\pi x}{2a} = \frac{1}{2} \left( \int_0^a dx \cos\frac{(m+n)\pi x}{2a} + \int_0^a dx \cos\frac{(m-n)\pi x}{2a} \right) \tag{84}$$

$$= \frac{a}{2} \left( \frac{\sin\frac{(m+n)\pi}{2}}{\frac{(m+n)\pi}{2}} + \frac{\sin\frac{(m-n)\pi}{2}}{\frac{(m-n)\pi}{2}} \right) \tag{85}$$

Let us define $m = n + k$ where $k$ can be any even integer, since $m$ and $n$ are both odd. With this substitution, the equation (85) becomes

$$\int_0^a dx \cos\frac{n\pi x}{2a}\cos\frac{m\pi x}{2a} = \frac{a}{2}\left(\frac{\sin(n+\frac{k}{2})\pi}{(n+\frac{k}{2})\pi} + \frac{\sin\frac{k\pi}{2}}{\frac{k\pi}{2}}\right) \tag{86}$$

$$= \frac{a}{2}\left(-\frac{\sin(\frac{k}{2})\pi}{(n+\frac{k}{2})\pi} + \frac{\sin\frac{k\pi}{2}}{\frac{k\pi}{2}}\right) \tag{87}$$

$$= \frac{a}{2}\left(\frac{\sin\frac{k\pi}{2}}{\frac{k\pi}{2}}\right) \tag{88}$$

$$= \begin{cases} \frac{a}{2} & k = 0\ (i.e., m = n) \\ 0 & k \neq 0\ (i.e., m \neq n) \end{cases} \tag{89}$$

In other words, one can state

$$\int_0^a dx \cos\frac{n\pi x}{2a}\cos\frac{m\pi x}{2a} = \frac{a}{2}\,\delta_{mn}. \tag{90}$$

From equations (eqn.need.proof.proved) and (83), one gets

$$0 = \sqrt{\frac{a}{2}}\sum_{n=1,3,\ldots} b_n\delta_{mn}, \tag{91}$$

from which we obtain

$$b_m = 0 \tag{92}$$

where $m = 1, 3, 5, \ldots$. Thus we have proved that each and every term in the parenthesis of the summation in the equation (82) must be zero, i.e.,

$$\dot{\alpha}_n(t) = -\left(1 + \frac{n^2\pi^2}{4a^2}\right)\alpha_n(t). \tag{93}$$

Solution for this equation is given by

$$\alpha_n(t) = \alpha_n(0)\exp\left[-\left(1 + \frac{n^2\pi^2}{4a^2}\right)t\right]. \tag{94}$$

The coefficients, $\alpha_n(0)$, are determined through the initial condition,

$$v(x,0) = \sum_{n=1,3,\ldots}\alpha_n(0)\sqrt{\frac{2}{a}}\cos\frac{n\pi}{2a}. \tag{95}$$

By integrating both sides with $\cos(m\pi x/2a)$, one gets

$$\int_0^a v(x,0)\cos\frac{m\pi x}{2a}dx = \sum_{n=1,3,\ldots}\sqrt{\frac{2}{a}}\alpha_n(0)\int_0^a \cos\frac{m\pi x}{2a}\cos\frac{n\pi x}{2a}dx. \tag{96}$$

From the equations (90) and (96), we obtain

$$1 = \sqrt{\frac{a}{2}} \sum_{n=1,3,\ldots} \alpha_n(0)\delta_{nm}. \tag{97}$$

We can now write the coefficient $\alpha_m(0)$ as

$$\alpha_m(0) = \sqrt{\frac{2}{a}}. \tag{98}$$

Consequently using equations (98), (94) and (75), one obtains the complete solution,

$$v(x,t) = \frac{2}{a} \sum_{n=1,3,\ldots} \cos\frac{n\pi x}{2a} \exp\left[-\left(1+\frac{n^2\pi^2}{4a^2}\right)t\right]. \tag{99}$$

$v(x,t)$ is plotted in figure 6.

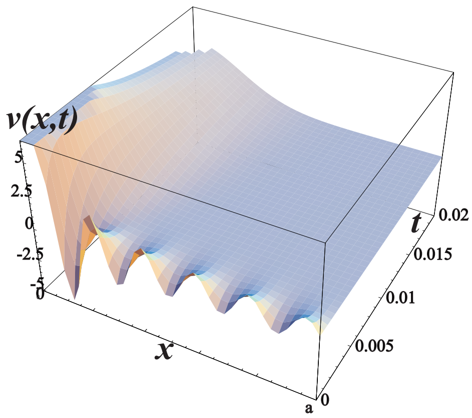


Figure 6: $v_{1,2,3,4}(x,t)$ vs. $x,t$. Higher modes, in other words, narrower cosines, die off quickly by the time, so the initial Dirac delta function broadens while its magnitude decreases (see equation(99)).