PHYS483 HW VI by Deyu Lu

Problem 1. Hamiltonian and total charge of free Dirac field

$$\mathcal{L}_D = \bar{\psi}\left(\frac{i}{2}\gamma^\mu \overleftrightarrow{\partial_\mu} - mI\right)\psi$$

$$= \bar{\psi}\left[\frac{i}{2}\gamma^\mu\left(\overrightarrow{\partial_\mu} - \overleftarrow{\partial_\mu}\right) - mI\right]\psi$$

$$\pi_\psi = \frac{\partial \mathcal{L}_D}{\partial \dot{\psi}} = \bar{\psi}\frac{i}{2}\gamma^0, \quad \pi_{\bar{\psi}} = \frac{\partial \mathcal{L}_D}{\partial \dot{\bar{\psi}}} = \frac{i}{2}\gamma^0\psi$$

$$\mathcal{H} = \pi_\psi \dot{\psi} - \dot{\bar{\psi}}\pi_{\bar{\psi}} - \mathcal{L}$$

$$= \frac{i}{2}\bar{\psi}\gamma^0(\partial_t\psi) - \frac{i}{2}(\partial_t\bar{\psi})\gamma^0\psi - \bar{\psi}\left(\frac{i}{2}\gamma^\mu\overleftrightarrow{\partial_\mu} - mI\right)\psi$$

Recall the Dirac Equation:

$$(i\overrightarrow{\not\partial} - m)\psi = 0 \quad \text{and} \quad \bar{\psi}(-i\overleftarrow{\not\partial} - m) = 0$$

We have

$$\mathcal{H} = \bar{\psi}\frac{i}{2}\gamma^0(\partial_t\psi) - \frac{i}{2}(\partial_t\bar{\psi})\gamma^0\psi$$

$$= i\bar{\psi}\gamma^0\partial_t\psi$$

$$= i\psi^\dagger\partial_t\psi$$

$$H = \int d^3x :\psi^\dagger(x)\, i\partial_t\psi(x):$$

$$\hat{H} = \int d^3x \; :\psi^+(x)\, i\partial_t \psi(x):$$

$$= \int d^3x \frac{1}{(2\pi)^3} \int d^3k \int d^3k' \; \frac{m}{k_0}\frac{m}{k_0'}$$

$$: \sum_{\alpha\beta} \left[b_\alpha^+(k)\bar{u}^\alpha(k) e^{ikx} + d_\alpha(k)\bar{V}^\alpha(k) e^{-ikx} \right] \cdot$$

$$\left[b_\beta(k') u^\beta(k') k_0' e^{-ik'x} + d_\beta^+(k') V^\beta(k) (-k_0') e^{ik'x} \right]:$$

Recall the integrals;

$$\int d^3x \; e^{\pm i(k-k')x} = (2\pi)^3 \delta(\vec{k}-\vec{k}') \quad \text{and also}$$

$\bar{u}^\alpha V^\beta = 0$ (Ryder, Eq 2.138)

and add (-1) when reorder $d_\alpha d_\beta^+$. We have

$$\hat{H} = \int d^3k \; \frac{m^2}{k_0} \cdot \sum_{\alpha\beta} \left(b_\alpha^+ b_\beta \bar{u}^\alpha u^\beta + d_\beta^+ d_\alpha \bar{V}^\alpha V^\beta \right)$$

Exploit the property: $\bar{u}^\alpha u^\beta = \bar{V}^\alpha V^\beta = \delta_{\alpha\beta}\frac{k_0}{m}$.
(See Ryder Eq 2.139).

$$\hat{H} = \int d^3k \; \frac{m}{k_0} k_0 \sum_\alpha \left[b_\alpha^+(\vec{k}) b_\alpha(\vec{k}) + d_\alpha^+(\vec{k})\, d_\alpha(\vec{k}) \right]$$

Now we repeat the calculation on Q

$$Q = \int d^3x \; :j_0(x):$$

$$= \int d^3x \; :\psi^+(x)\,\psi(x):$$

$$= \int d^3x \frac{1}{(2\pi)^3} \int d^3k \int d^3k' \, \frac{m}{k_0} \frac{m}{k_0'}$$

$$:\sum_{\alpha\beta} \left[b_\alpha^+(k)\bar{u}^\alpha(k) e^{ikx} + d_\alpha(k)\bar{v}^\alpha(k) e^{-ikx} \right] \cdot \left[b_\beta(k') u^\beta(k') e^{-ik'x} + d_\beta^+(k') v^\beta(k') e^{ik'x} \right]:$$

$$= \int d^3k \, \frac{m^2}{k_0^2} \sum_{\alpha\beta} \left(b_\alpha^+ b_\beta \, \bar{u}^\alpha u^\beta - d_\beta^+ d_\alpha \, \bar{v}^\alpha v^\beta \right)$$

$$= \int d^3k \, \frac{m}{k_0} \sum_\alpha \left(b_\alpha^+(\vec{k}) b_\alpha(\vec{k}) - d_\alpha^+(\vec{k}) d_\alpha(\vec{k}) \right)$$

problem 2 EOM for Dirac particle with EM field

$$\mathcal{L} = \mathcal{L}_\gamma + \mathcal{L}_D + \mathcal{L}_I$$

$$\mathcal{L}_\gamma = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \quad \mathcal{L}_D = \sum_f \overline{\psi^{(f)}}(x) \left(\frac{i}{2} \gamma^\mu \overleftrightarrow{\partial_\mu} - m_f I\right) \psi^f(x)$$

$$\mathcal{L}_I = -e \sum_f \overline{\psi^{(f)}(x)} \gamma^\mu \psi^f(x) A_\mu(x).$$

a). $$S_D + S_I = \int d^4x \quad \bar{\psi} \left(\frac{i}{2} \gamma^\mu \overleftrightarrow{\partial_\mu} - m_f I\right) \psi - e \bar{\psi} \gamma^\mu \psi A_\mu$$

$$= \int d^4x \quad i \bar{\psi} \not{\partial}_\mu \psi - m_f \bar{\psi} \psi - e \bar{\psi} \gamma^\mu \psi A_\mu$$

$$\bar{\psi} \to \bar{\psi} + \delta \bar{\psi}$$

$$\delta S_D + \delta S_I = \int d^4x \quad i \delta \bar{\psi} \not{\partial}_\mu \psi - m_f \delta \bar{\psi} \psi - e \delta \bar{\psi} \gamma^\mu \psi A_\mu$$

$$= 0$$

we reach: $(i \not{\partial}_\mu - m_f I) \psi^f(x) = e \gamma^\nu \psi^f(x) A_\nu(x)$

b) $$\mathcal{L}_\gamma + \mathcal{L}_I = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - e \bar{\psi} \gamma^\mu \psi A_\mu$$

$$\mathcal{L}_\gamma = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$$= -\frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$$

$$= -\frac{1}{2} (\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu)$$

• Euler Lagrangian equation

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} - \frac{\partial \mathcal{L}}{\partial(A_\nu)} = 0$$

$$\mathcal{L}_\gamma = -\frac{1}{2}\left(\partial_\mu A_\nu \, \partial_\rho A_\sigma \, g^{\rho\mu} g^{\sigma\nu} - \partial_\mu A_\nu \, \partial_\rho A_\sigma \, g^{\rho\nu} g^{\sigma\mu} \right)$$

$$\frac{\partial \mathcal{L}_\gamma}{\partial(\partial_\mu A_\nu)} = -\frac{1}{2}\left[\partial^\mu A^\nu + \partial^\mu A^\nu - \partial^\nu A^\mu - \partial^\nu A^\mu \right)$$

$$= - F^{\mu\nu}$$

$$\frac{\partial \mathcal{L}_I}{\partial A_\nu} = -e\bar{\psi}\gamma^\nu \psi$$

EOM reads:

$$-\partial_\mu F^{\mu\nu} + e\bar{\psi}\gamma^\nu\psi = 0.$$

Remember $\partial_\mu A^\mu = 0$

$$\partial_\mu F^{\mu\nu} = \partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) = (\partial_\mu \partial^\mu) A^\nu - \cancel{\partial^\nu \partial_\mu A^\mu}$$

Finally.

$$\partial_\mu \partial^\mu A^\nu = e\bar{\psi}\gamma^\nu\psi$$

Problem 3: Feynman propagator of EM field

$$A_\mu(x) = \frac{1}{\sqrt{(2\pi)^3}} \int \frac{d^3k}{2\omega_k} \sum_{\lambda=0}^{3} \varepsilon_\mu^\lambda(k) \left[a^\lambda(k) e^{-ikx} + a^{\lambda+}(k) e^{ikx} \right]$$

$$[a^\lambda(k), a^{\lambda'+}(k')] = -g^{\lambda\lambda'} 2\omega_k \delta(\vec{k}-\vec{k}')$$

$$\varepsilon^\lambda \cdot \varepsilon^{\lambda'} = g^{\lambda\lambda'}$$

Now calculate the propagator:

$$\langle 0 | T[A^\mu(x) A^\nu(y)] | 0 \rangle$$

$$= \frac{1}{(2\pi)^3} \int \frac{d^3k_1}{2\omega_{k_1}} \int \frac{d^3k_2}{2\omega_{k_2}} \sum_{\lambda_1} \sum_{\lambda_2} \varepsilon^{\lambda_1 \mu}(k_1) \varepsilon^{\lambda_2 \nu}(k_2)$$

$$\boxed{\begin{aligned} &\langle 0 | T[(a^{\lambda_1}(k_1) e^{-ik_1 x} + a^{\lambda_1 +}(k_1) e^{ik_1 x}) \\ &\quad (a^{\lambda_2}(k_2) e^{-ik_2 x} + a^{\lambda_2 +}(k_2) e^{ik_2 y}) | 0 \rangle \end{aligned}}$$

$$\Downarrow$$

$$\langle 0 | T[\cdots] | 0 \rangle$$

$$\langle 0 | T[\cdots] | 0 \rangle = \theta(t_x - t_y) (-g^{\lambda_1\lambda_2} 2\omega_k \delta(\vec{k}-\vec{k}')) e^{-ik\cdot(x-y)}$$

$$- \theta(t_y - t_x) (-g^{\lambda_1\lambda_2}) 2\omega_k \delta(\vec{k}-\vec{k}') e^{-ik(y-x)}$$

$$= -g^{\lambda_1\lambda_2} \cdot 2\omega_k \delta(\vec{k}-\vec{k}') \left[\theta(t_x - t_y) e^{-ik(x-y)} + \theta(t_y - t_x) e^{-ik(y-x)} \right]$$

$\langle 0 | T[A^\mu(x) A^\nu(y)] | 0 \rangle$

$$= \frac{-1}{(2\pi)^3} \int \frac{d^3k}{2\omega_k} \sum_{\lambda_1 \lambda_2} \varepsilon^{\lambda_1 \mu} \varepsilon^{\lambda_2 \nu} g_{\lambda_1 \lambda_2} \left[\theta(t_x - t_y) e^{+ik(-x+y)} + \theta(t_y - t_x) e^{ik(x-y)} \right]$$

$$\sum_{\lambda_1, \lambda_2} \varepsilon^{\lambda_1 \mu} \varepsilon^{\lambda_2 \nu} g^{\lambda_1 \lambda_2} = \sum_{\lambda_1 \lambda_2} \varepsilon^{\lambda_1 \mu} \varepsilon^{\lambda_2 \nu} g_{\lambda_1 \lambda_2}$$

$$= \sum_\lambda \varepsilon_\lambda{}^\mu \varepsilon^{\lambda \nu}$$

$$= g^{\mu\nu}$$

$\langle 0 | T[A^\mu(x) A^\nu(y)] | 0 \rangle$

$$= \frac{-1}{(2\pi)^3} \int \frac{d^3k}{2\omega_k} g^{\mu\nu} \left[\theta(t_x - t_y) e^{ik(y-x)} + \theta(t_y - t_x) e^{ik(x-y)} \right]$$

Substitute $\vec{k} \to -\vec{k}$ in the second term.

$$= \frac{-g^{\mu\nu}}{(2\pi)^3} \int \frac{d^3k}{2\omega_k} e^{+i\vec{k}\cdot(\vec{x}-\vec{y})} \left[\theta(t_x - t_y) e^{i\omega_k(t_y - t_x)} + \theta(t_y - t_x) e^{i\omega_k(t_x - t_y)} \right] \qquad (*)$$

Define $t \equiv t_x - t_y$.

$$\tilde{F}(\omega) = \int_{-\infty}^{\infty} dt \, f(t) \, e^{i\omega t} \iff f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \, \tilde{F}(\omega) \, e^{-i\omega t}$$

$$\int_{-\infty}^{\infty} dt \left[\theta(t) e^{-i\omega_k t} + \theta(-t) e^{i\omega_k t}\right] e^{i\omega t}$$

$$= \int_{-\infty}^{\infty} dt \left[\theta(t) e^{i(\omega-\omega_k)t} + \theta(-t) e^{i(\omega+\omega_k)t}\right]$$

$$= \frac{-1}{i(\omega + i\epsilon - \omega_k)} + \frac{1}{i(\omega - i\epsilon + \omega_k)}$$

$$= \frac{1}{i}\left[\frac{1}{\omega + \omega_k - i\epsilon} - \frac{1}{\omega - \omega_k + i\epsilon}\right]$$

$$= \frac{1}{i} \quad \frac{-2\omega_k}{\omega^2 - \omega_k^2 + i\epsilon + O(\epsilon^2)}$$

$$= i \quad \frac{2\omega_k}{\omega^2 - \omega_k^2 + i\epsilon}$$

Put it back to (*):

$$\langle 0 | T[A^\mu(x) A^\nu(y)] | 0 \rangle = \frac{i}{(2\pi)^4} \int d^4k \, e^{-ik(x-y)} \frac{-g^{\mu\nu}}{k^{0\,2} - \omega_k^2 + i\epsilon}$$

$$= \frac{i}{(2\pi)^4} \int d^4k \, e^{-ik(x-y)} \frac{-g^{\mu\nu}}{k^2 + i\epsilon}$$

The last derivation makes use of $\omega_k^2 = |\vec{k}|^2$.

Problem 4: Feynman propagator of Dirac particle

$$\psi(x)=\int\frac{d^3k}{\sqrt{(2\pi)^3}}\,\frac{m}{\omega_k}\sum_\alpha\left[b_\alpha(\vec{k})\,u^\alpha(\vec{k})e^{-ikx}+d_\alpha^+(\vec{k})\,v^\alpha(\vec{k})e^{ikx}\right]$$

$$\psi^+(x)=\int\frac{d^3k}{\sqrt{(2\pi)^3}}\,\frac{m}{\omega_k}\sum_\alpha\left[b_\alpha^+(\vec{k})\,\bar{u}^\alpha(\vec{k})e^{ikx}+d_\alpha(\vec{k})\,\bar{v}^\alpha(\vec{k})e^{-ikx}\right]$$

$$\{b_\alpha(\vec{k}),\,b_\beta^+(\vec{k}')\}=\frac{k_0}{m}\delta(\vec{k}-\vec{k}')\,\delta_{\alpha\beta}$$

$$\{d_\alpha(\vec{k}),\,d_\beta^+(\vec{k}')\}=\frac{k_0}{m}\delta(\vec{k}-\vec{k}')\,\delta_{\alpha\beta}$$

$$\langle 0|T[\psi(x)\,\overline{\psi(y)}]|0\rangle$$

$$=\frac{1}{(2\pi)^3}\int d^3k_1\,\frac{m}{\omega_{k_1}}\int d^3k_2\,\frac{m}{\omega_{k_2}}\sum_{\alpha\beta}\langle 0|T[\Big(b_\alpha(k_1)\,u^\alpha(k_1)\,e^{-ik_1x}$$

$$+d_\alpha^+(k_1)\,v^\alpha e^{ik_1x}\Big)\Big(b_\beta^+(k_2)\,\bar{u}^\beta(k_2)\,e^{ik_2y}+d_\beta(k_2)\,\bar{v}^\beta(k_2)\,e^{-ik_2y}\Big)]|0\rangle$$

Be careful that exchange fermi operators give extra -1,

$$\langle 0|T[\cdots]|0\rangle$$

$$=\theta(t_x-t_y)\,u^\alpha(k_1)\,\bar{u}^\beta(k_2)\,e^{-ik_1x}e^{ik_2y}\,\gamma^0\,\delta(\vec{k}_1-\vec{k}_2)\,\delta_{\alpha\beta}\,\frac{\omega_{k_1}}{m}$$

$$-\theta(t_y-t_x)\,v^\alpha(k_1)\,\bar{v}^\beta(k_2)\,e^{ik_1x}e^{-ik_2y}\,\gamma^0\,\delta(k_1-k_2)\,\delta_{\alpha\beta}\,\frac{\omega_{k_1}}{m}$$

$\langle 0|T[\psi(x)\,\bar{\psi}(y)]|0\rangle$

$$= \frac{1}{(2\pi)^3}\int d^3k\,\frac{m}{\omega_k}\sum_\alpha\Big[\theta(t_x-t_y)e^{-ik(x-y)}u^\alpha\bar{u}^\beta\gamma^0 - \theta(t_y-t_x)e^{ik(x-y)}v^\alpha\bar{v}^\beta\gamma^0\Big]$$

We will show later

$$\sum_\alpha u^\alpha\bar{u}^\alpha\gamma^0 = \frac{\not{k}+m}{2m}, \quad \text{and}$$

$$\sum_\alpha v^\alpha\bar{v}^\alpha\gamma^0 = \frac{\not{k}-m}{2m}.$$

$\langle 0|T[\psi(x)\,\bar{\psi}(y)]|0\rangle$

$$= \frac{1}{(2\pi)^3}\int\frac{d^3k}{2\omega_k}\,[\theta(t_x-t_y)(\not{k}+m)e^{-ik(x-y)} - \theta(t_y-t_x)(\not{k}-m)e^{ik(x-y)}]$$

$$= \frac{1}{(2\pi)^3}\int\frac{d^3k}{2\omega_k}\,e^{i\vec{k}\cdot(\vec{x}-\vec{y})}[\theta(t)(\not{k}+m)e^{-i\omega_k t} + \theta(-t)(\not{k}+m)e^{i\omega_k t}]$$

$$\int_{-\infty}^{\infty}dt\,[\theta(t)\,e^{-i\omega_k t} + \theta(-t)\,e^{i\omega_k t}] = \frac{2\omega_k i}{\omega^2-\omega_k^2+i\epsilon}$$

Then we have

$$\langle 0|T[\psi(x)\,\bar{\psi}(y)]|0\rangle = \frac{i}{(2\pi)^4}\int d^4k\; e^{-ik(x-y)}\,\frac{\not{k}+mI}{k_0^2-\omega_k^2+i\epsilon}$$

Remember: $\omega_k^2 = m^2+|\vec{k}|^2$, then

$$k_0^2-\omega_k^2 = k_0^2-m^2-|\vec{k}|^2 = k^2-m^2$$

$$\langle 0|T[\psi(x),\bar{\psi}(y)]|0\rangle = \frac{i}{(2\pi)^4}\int d^4k\; e^{-ik(x-y)}\,\frac{\not{k}+mI}{k^2-m^2+i\epsilon}$$

The derivations above are indeed tedious.
A much nicer way to prove it is to show

$$(i\not{\partial} - m)\langle 0|T[\psi(x)\bar{\psi}(y)]|0\rangle = i\delta^4(x-y),$$

and then do the Fourier transformation.

Verify
$$\begin{cases} \sum_\alpha u^\alpha \bar{u}^\alpha \gamma^0 = \dfrac{\not{p}+m}{2m} & \text{— (1)} \\ \sum_\alpha v^\alpha \bar{v}^\alpha \gamma^0 = \dfrac{\not{p}-m}{2m} & \text{— (2)} \end{cases}$$

Let us only prove the first Eq. The second one can be proved similarly.

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} \sqrt{\frac{E+m}{2m}}$$

$$u_2 = \begin{pmatrix} 0 \\ 1 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} \sqrt{\frac{E+m}{2m}}$$

$$v_1 = \begin{pmatrix} -\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} \sqrt{\frac{E+m}{2m}}$$

$$v_2 = \begin{pmatrix} -\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} \sqrt{\frac{E+m}{2m}}$$

(See HW II)

$$\frac{\not{p}+m}{2m} = \frac{p_\mu \gamma^\mu + m}{2m}$$

$$= \frac{\begin{pmatrix} E & \\ & -E \end{pmatrix} + \begin{pmatrix} m & \\ & m \end{pmatrix} + \begin{pmatrix} 0 & -\vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & 0 \end{pmatrix}}{2m}$$

$$= \frac{1}{2m} \begin{pmatrix} m+E & -\vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & m-E \end{pmatrix}$$

$$u_1 \bar{u}_1 \gamma^0 + u_2 \bar{u}_2 \gamma^0$$

$$= \begin{pmatrix} 1 \\ 0 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} \left(1 . 0 \;\; -(1 . 0) \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \right) \frac{E+m}{2m}$$

$$+ \begin{pmatrix} 0 \\ 1 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} \left(0 \;\; 1 \;\; -(0 \;\; 1) \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \right) \frac{E+m}{2m}$$

$$= \begin{pmatrix} 1 & 0 & -(1 . 0)\frac{\vec{\sigma}\cdot\vec{p}}{E+m} \\ 0 & 0 & 0 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\begin{pmatrix} 1 \\ 0 \end{pmatrix} & 0 & -\left(\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\right)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\left(\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\right) \end{pmatrix} \left(\frac{E+m}{2m}\right)$$

$$+ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -(0 . 1)\frac{\vec{\sigma}\cdot\vec{p}}{E+m} \\ 0 & \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\begin{pmatrix} 0 \\ 1 \end{pmatrix} & -\left(\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\right)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\left(\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\right) \end{pmatrix} \left(\frac{E+m}{2m}\right)$$

$$= \frac{1}{2m}\begin{pmatrix} E+m & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -\frac{p^2}{E+m} \end{pmatrix}$$

$$= \frac{1}{2m}\begin{pmatrix} E+m & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & m-E \end{pmatrix}$$

$$= \frac{\not{p}+m}{2m}$$