1. $J^{\mu}{}_{\rho} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Phi_\rho - \Theta^{\mu}{}_{\nu} \underline{X}^{\nu}{}_{\rho}$

Spacial transformation: $\vec{x} \to \vec{x} + \vec{a}$,

$\Phi_\rho = 0$ and $\underline{X}^{\nu}{}_{\rho} = \delta^{\nu}{}_{\rho}$, $\rho = 1.2.3$

The time independent conservative:

$J^{0}{}_{i} = -\theta^{0}{}_{\mu} \delta^{\mu}{}_{i} = -\theta^{0}{}_{i}$

By definition: $\Theta^{0}{}_{i} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} \partial_i \phi \cancel{- \delta^{0}{}_{i} \mathcal{L}}$,

with $\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2$.

$J^{0}{}_{i} = -\frac{\partial \mathcal{L}}{\partial(\dot{\phi})} \partial_i \phi = -\dot{\phi} \partial_i \phi$

$\vec{P} = -\int_{V_\infty} d^3x \; \dot{\phi} \nabla \phi$.

2. $\int_\Omega d^4x \; \partial_\mu [\phi_n^* \overleftrightarrow{\partial^\mu} \phi_m]$

$$= \int_\Omega d^4x \; \partial_\mu [\, \phi_n^* \partial^\mu \phi_m - (\partial^\mu \phi_n^*)\, \phi_m \,]$$

$$= \int_\Omega d^4x \; (\partial_\mu \phi_n^*)(\partial^\mu \phi_m) + \phi_n^* \, \partial_\mu \partial^\mu \phi_m$$
$$- (\partial_\mu \partial^\mu \phi_n^*)\phi_m - (\partial^\mu \phi_n^*)(\partial_\mu \phi_m)$$

We then show $(\partial_\mu \phi_n^*)(\partial^\mu \phi_m) = (\partial^\mu \phi_n^*)(\partial_\mu \phi_m)$:

$$(\partial_\mu \phi_n^*)(\partial^\mu \phi_m) = (g_{\mu\nu} \partial^\nu \phi_n^*)(g^{\mu\rho} \partial_\rho \phi_m)$$
$$= \delta^\rho{}_\nu (\partial^\nu \phi_n^*)(\partial_\rho \phi_m)$$
$$= (\partial^\mu \phi_n^*)(\partial_\mu \phi_m)$$

Then: $\int_\Omega d^4x \; \partial_\mu [\phi_n^* \overleftrightarrow{\partial^\mu} \phi_m]$

$$= \int_\Omega d^4x \;\; \phi_n^* \partial_\mu \partial^\mu \phi_m - (\partial_\mu \partial^\mu \phi_n^*)\, \phi_m$$
$$+ \; m^2 \phi_n^* \phi_m - m^2 \phi_n^* \phi_m$$

$$= \int_\Omega d^4x \; \left\{ \phi_n^* \, (\partial_\mu \partial^\mu + m^2)\, \phi_m - \left[(\partial_\mu \partial^\mu + m^2)\, \phi_n^* \right] \phi_m \right\}$$

3. $f_{\vec{k}} = \dfrac{1}{\sqrt{(2\pi)^3 2E(k)}} e^{-ik_\mu x^\mu}$, $E(k) = k_0 = \sqrt{|\vec{k}|^2 + m^2}$.

$E(\vec{k}) = E(-\vec{k})$

(1) $\langle f_{\vec{k}} | f_{\vec{k}'} \rangle$

$= \int d^3x \; f_{\vec{k}}^* (i\overleftrightarrow{\partial_0}) f_{\vec{k}'}$

$= \dfrac{1}{(2\pi)^3} \dfrac{1}{\sqrt{2E(k) \cdot 2E(k')}} \int d^3x \; (k_0 + k_0') e^{-i(k_\mu' - k_\mu)x^\mu}$

$= \dfrac{1}{\sqrt{2E(k) \cdot 2E(k')}} (k_0 + k_0') e^{-i(k_0' - k_0)x^0} \delta(\vec{k} - \vec{k}')$

$= \dfrac{1}{2E(k)} \cdot 2k_0 \, \delta(\vec{k} - \vec{k}')$

$= \delta(\vec{k} - \vec{k}')$

(2) $\langle f_{\vec{k}} | f_{\vec{k}'}^* \rangle$

$= \int d^3x \; f_{\vec{k}}^* (i\overleftrightarrow{\partial_0}) f_{\vec{k}'}^*$

$= \text{Const} \cdot \int d^3x \; (-k_0' + k_0) \, e^{i(k_\mu' + k_\mu)x^\mu}$

$= 0$

(3) $\langle f_{\vec{k}}^* | f_{\vec{k}'} \rangle$

$= \int d^3x \; f_{\vec{k}} (i\overleftrightarrow{\partial_0}) f_{\vec{k}'}$

$= \text{Const} \cdot \int d^3x \; (k_0' - k_0) \, e^{-i(k_\mu' + k_\mu)x^\mu}$

$= 0$

4. $g(x) = \int d^3\vec{k}\ f_{\vec{k}}(x)\ \alpha(\vec{k})$ with
$\alpha(\vec{k}) = \langle f_{\vec{k}}(x) \mid g(x) \rangle_{x^0}$

① To prove the completeness of the basis function, we need to confine $g(x)$ as physical observables, which means:

$$g(x) = \int d^4k\ f_{\vec{k}}(x)\ \alpha(\vec{k})\ \delta(k^0 - \sqrt{|\vec{k}|^2 + m^2}),\ \text{i.e.}$$

$$\omega \equiv \omega(\vec{k}) = \sqrt{|\vec{k}|^2 + m^2}.$$

One can postulate it by the following. Any measurement is based on plane wave and the same dispersion relation should always apply.

② If we admit $f_{\vec{k}}(x)$ is complete for certain $g(x)$, then it is straight forward to find the coefficients.

$$\langle f_{\vec{k}}(x) \mid g(x) \rangle = \int d^3\vec{k}'\ \alpha(\vec{k}')\ \langle f_{\vec{k}} \mid f_{\vec{k}'} \rangle$$
$$= \int d^3\vec{k}'\ \alpha(\vec{k}')\ \delta(\vec{k} - \vec{k}')$$
$$= \alpha(\vec{k})$$

5. Show the commutation rule $i[\hat{p}_i, \hat{q}_j] = \delta_{ij}$ is consistant with $\hat{q}_j$ multiplicative and $\hat{p}_j = -i\partial_j$.

First we want to find a proper definition for $\hat{p}_j$.

① $\hat{q}_j |\psi(q)\rangle = q_j |\psi(q)\rangle$

② $|\psi(q)\rangle = \int \frac{d^3\vec{p}}{(2\pi)^3} e^{i\vec{p}\cdot\vec{q}} |\psi(\vec{p})\rangle$

③ $\hat{p}_j |\psi(q)\rangle = \int \frac{d^3\vec{p}}{(2\pi)^3} e^{i\vec{p}\cdot\vec{q}} \hat{p}_j |\psi(\vec{p})\rangle$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} e^{i\vec{p}\cdot\vec{q}} p_j |\psi(\vec{p})\rangle$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} p_j e^{i\vec{p}\cdot\vec{q}} |\psi(\vec{p})\rangle$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} (-i\partial_{q_j}) e^{i\vec{p}\cdot\vec{q}} |\psi(\vec{p})\rangle$$

This means in $\vec{q}$ space, $\hat{p}_j$ is defined as $-i\partial_{q_j}$.

Next we show it consistant with the commutation rule.

$i[\hat{p}_i, \hat{q}_j] = i[-i\partial_i, \hat{q}_j] = \delta_{ij}$.

6. (1) $\partial_\mu \hat{F} = i[P_\mu, F]$, where $p = [H, \vec{p}\,]$.

We can postulate it by making analogy to classical mechanics.

$$\frac{dF(t, q, p)}{dt} = \frac{\partial F(t, q, p)}{\partial t} + \{H, F\}$$

If F doesn't depend on t explicitly, we have

$$\frac{dF}{dt} = \{H, F\}.$$

The correspondence principle applies:

$$\{\ \} \Rightarrow i[\ \ ].$$

By changing variables into operators:

$$\partial_\mu \hat{F} = i[\hat{P}_\mu, \hat{F}].$$

Alternatively, we can use the result of Prob. 8, namely,

$$\begin{cases} \hat{H} = \frac{1}{2}\int d^3\vec{k}\ E(\vec{k})\,[\hat{a}^+(\vec{k})\,\hat{a}(\vec{k}) + \hat{a}(\vec{k})\,\hat{a}^+(\vec{k})] \\ \hat{\vec{P}} = \frac{1}{2}\int d^3\vec{k}\ \vec{k}\ [\hat{a}^+(\vec{k})\,\hat{a}(\vec{k}) + \hat{a}(\vec{k})\,\hat{a}^+(\vec{k})]. \end{cases}$$

We can then prove (2) first.

$$|\varphi(x)\rangle = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik_\mu x^\mu}\, |\varphi(k)\rangle$$

$$e^{-i\hat{p}_\mu (x'^\mu - x^\mu)}\, |\varphi(x)\rangle$$

$$= e^{-i\hat{p}_\mu (x'^\mu - x^\mu)} \int \frac{d^4\vec{k}}{(2\pi)^4}\, e^{-ik_\mu x^\mu}\, |\varphi(k)\rangle$$

$$= \int \frac{d^4\vec{k}}{(2\pi)^4}\, e^{-ik_\mu x'^\mu}\, |\varphi(k)\rangle$$

$$= |\varphi(x')\rangle$$

$$\langle \varphi(x')|\, F(x^\mu)\, |\varphi(x')\rangle$$

$$= \langle \varphi(x)|\, e^{i\hat{p}_\mu (x'^\mu - x^\mu)}\, F(x^\mu)\, e^{-i\hat{p}_\mu (x'^\mu - x^\mu)}\, |\varphi(x)\rangle$$

As the physics in Schrödinger picture and Heisenberg picture is equivallent, we conclude

$$F(x'^\mu) = e^{i\hat{p}_\mu (x'^\mu - x^\mu)}\, F(x^\mu)\, e^{-i\hat{p}_\mu (x'^\mu - x^\mu)}$$

Once (2) is proved, it is easy to show (1).

$$F(x^\mu) = e^{i\hat{P}_\mu (x^\mu - x_0^\mu)} F(x_0^\mu)\, e^{-i\hat{P}_\mu (x^\mu - x_0^\mu)}$$

$$\begin{aligned}
\partial_\mu F(x^\mu) &= e^{i\hat{P}_\mu (x^\mu - x_0^\mu)} (i\hat{P}_\mu) F(x_0^\mu) e^{-i\hat{P}_\mu (x^\mu - x_0^\mu)} \\
&\quad - e^{i\hat{P}_\mu (x^\mu - x_0^\mu)} F(x_0^\mu) (i\hat{P}_\mu) e^{-i\hat{P}_\mu (x^\mu - x_0^\mu)} \\
&= i\hat{P}_\mu F(x^\mu) - F(x^\mu)\, i\hat{P}_\mu \\
&= i[\hat{P}_\mu, F(x^\mu)]
\end{aligned}$$

(2) Given $\partial_\mu \hat{F} = i[\hat{P}_\mu, F]$ satisfied, we can prove (2) by exploiting the identity:

$$e^{\hat{A}} \hat{B} e^{-\hat{A}} = \hat{B} + [\hat{A}, \hat{B}] + \frac{1}{2!}[\hat{A}[\hat{A}, B]] + \cdots$$

$$\begin{aligned}
&e^{i\hat{P}_\mu (x'^\mu - x^\mu)} F(x^\mu)\, e^{-i\hat{P}_\mu (x'^\mu - x^\mu)} \\
&= F(x^\mu) + i(x'^\mu - x^\mu)[\hat{P}_\mu, F] + \frac{1}{2!} i^2 (x'^\mu - x^\mu)^2 [\hat{P}_\mu, [\hat{P}_\mu, F]] \\
&\quad + \cdots \\
&= F(x^\mu) + (x'^\mu - x^\mu)\partial_\mu F + \frac{1}{2!}(x'^\mu - x^\mu)^2 \partial_\mu^2 F + \cdots \\
&= F(x'_\mu)
\end{aligned}$$

7. $\hat{a}(\vec{k}) = i\int d^3x \; f^*_{\vec{k}}(x) \overleftrightarrow{\partial_0} \hat{\varphi}(x)$ and

$\hat{a}^+(\vec{k}) = -i\int d^3x \; f_{\vec{k}}(x) \overleftrightarrow{\partial_0} \hat{\varphi}(x)$.

$[\hat{a}(\vec{k}), \hat{a}(\vec{k}')]\; k^0 = k'^0$

$= i^2 \int d^3x \int d^3x' \; [f^*_{\vec{k}}(x) \overleftrightarrow{\partial_0} \varphi(x), \; f^*_{\vec{k}'}(x') \overleftrightarrow{\partial_0'} \hat{\varphi}(x')]$

$= i^2 \int d^3x \int d^3x' \; f^*_{\vec{k}}(x) f^*_{\vec{k}'}(x') \overleftrightarrow{\partial_0} \overleftrightarrow{\partial_0'} [\varphi(x), \varphi(x')]$

$f^*_{\vec{k}}(x) f^*_{\vec{k}'}(x') \overleftrightarrow{\partial_0} \overleftrightarrow{\partial_0'} [\varphi(x), \varphi(x')]$

$= f^*_{\vec{k}}(x) f^*_{\vec{k}'}(x') \cancel{[\pi(x), \pi(x')]} + f^*_{\vec{k}}(x) (-\partial_0' f^*_{\vec{k}'}(x')) [\pi(x), \varphi(x')]$

$- (\partial_0 f^*_{\vec{k}}(x)) f^*_{\vec{k}'}(x') [\varphi(x), \pi(x')] + \cancel{(\partial_0 f^*_{\vec{k}}(x))(\partial_0' f^*_{\vec{k}'}(x')) [\varphi(x), \varphi(x')]}$

$= f^*_k(x) \; (-\partial_0' f^*_{\vec{k}'}(x')) \frac{1}{i} \delta(x-x') + (\partial_0 f^*_{\vec{k}}(x)) f^*_{\vec{k}'}(x') \frac{1}{i} \delta(x-x')$

$[\hat{a}(\vec{k}), \hat{a}(\vec{k}')]$

$= \int d^3x \, d^3x' \; \delta(\vec{x}-\vec{x}') [-f^*_{\vec{k}}(x) (i\partial_0') f^*_{\vec{k}'}(x') + f^*_{\vec{k}'}(x') (i\partial_0 f^*_{\vec{k}}(x))]$

$= \int d^3x \, [f^*_{\vec{k}'}(x) \; i\overleftrightarrow{\partial_0} \; f^*_{\vec{k}}(x)]$

$= \langle f_{\vec{k}'}(x) \mid f^*_{\vec{k}}(x) \rangle$

$= 0$

· Similarly:

$$[\hat{a}^+(\vec{k}), \hat{a}^+(\vec{k}')] = -\langle f^*_{\vec{k}'}(x) \mid f_{\vec{k}}(x)\rangle = 0$$

8. (a) $\hat{H} = \frac{1}{2}\int d^3\vec{k}\ E(k)\,[\hat{a}^+(\vec{k})\hat{a}(\vec{k}) + \hat{a}(\vec{k})\hat{a}^+(\vec{k})]$

start from $\hat{H} = \int d^3x\ \{\frac{1}{2}[\hat{\pi}^2(x) + (\nabla\hat{\varphi}(x))^2 + m^2\hat{\varphi}^2(x)]\}$

$\hat{\varphi}(x) = \int d^3\vec{k}\ [f_{\vec{k}}(x)\hat{a}(\vec{k}) + f^*_{\vec{k}}\hat{a}^+(\vec{k})]$.

$\hat{\pi}(x) = \dot{\hat{\varphi}}(x) = \int d^3\vec{k}\ \left[(\partial_t f_{\vec{k}}(x))\hat{a}(\vec{k}) + (\partial_t f^*_{\vec{k}}(x))\hat{a}^+(\vec{k})\right]$

$\nabla\hat{\varphi}(x) = \int d^3\vec{k}\ [(\nabla f_{\vec{k}}(x))\hat{a}(\vec{k}) + (\nabla f^*_{\vec{k}}(x))\hat{a}^+(\vec{k})]$

$\begin{cases} \partial_t f_{\vec{k}}(x) = -ik_0 f_{\vec{k}}(x) \\ \partial_t f^*_{\vec{k}}(x) = ik_0 f^*_{\vec{k}}(x) \end{cases}$ $\quad\begin{cases} \nabla f_{\vec{k}}(x) = i\vec{k}\ f_{\vec{k}}(x) \\ \nabla f^*_{\vec{k}}(x) = -i\vec{k}\ f^*_{\vec{k}}(x) \end{cases}$

and $k_0^2 = |\vec{k}|^2 + m^2$.

① $\int d^3x\ \frac{1}{2}\hat{\pi}^2(x)$

$$= \frac{1}{2}\int d^3x \int d^3\vec{k}\int d^3\vec{k}'\ [-ik_0 f_{\vec{k}}\hat{a}(\vec{k}) + ik_0 f^*_{\vec{k}}\hat{a}^+(\vec{k})]\cdot [-ik_0' f_{\vec{k}'}\hat{a}(\vec{k}') + ik_0' f^*_{\vec{k}'} a^+(\vec{k}')]$$

$$\int d^3x\ f_{\vec{k}}(x) f_{\vec{k}'}(x) = \int d^3x \frac{1}{(2\pi)^3\sqrt{2E(k)\,2E(k')}}\, e^{-i(k_0+k_0')t}\, e^{i(\vec{k}+\vec{k}')\vec{x}}$$

$$= \delta(\vec{k}+\vec{k}')\frac{1}{2E(k)}\, e^{-2ik_0 t}$$

One can also prove:

$$\int d^3x \; f_{\vec{k}}(x) f^*_{\vec{k}'}(x) = \delta(\vec{k}-\vec{k}') \frac{1}{2E(k)}$$

$$\int d^3x \; f^*_{\vec{k}}(x) f_{\vec{k}'}(x) = \delta(\vec{k}-\vec{k}') \frac{1}{2E(k)}$$

$$\int d^3x \; f^*_{\vec{k}}(x) f^*_{\vec{k}'}(x) = \delta(\vec{k}+\vec{k}') \; e^{2ik_0t}$$

$$① = \frac{1}{2}\int \frac{d^3\vec{k}}{2E(k)} \Big[-k_0^2 e^{-2ik_0t} \hat{a}(\vec{k}) \hat{a}(-\vec{k}) + k_0^2 \hat{a}(\vec{k}) \hat{a}^+(\vec{k}) + k_0^2 \hat{a}^+(\vec{k}) \hat{a}(\vec{k}) - k_0^2 e^{2ik_0t} \hat{a}^+(\vec{k}) \hat{a}^+(-\vec{k}) \Big]$$

$$② \int d^3x \; \frac{1}{2} \left[\nabla\varphi(x)\right]^2$$

$$= \frac{1}{2}\int \frac{d^3\vec{k}}{2E(k)} \Big[|\vec{k}|^2 e^{-2ik_0t} \hat{a}(\vec{k}) \hat{a}(-\vec{k}) + |\vec{k}|^2 \hat{a}(\vec{k}) \hat{a}^+(\vec{k}) + |\vec{k}|^2 \hat{a}^+(\vec{k}) \hat{a}(\vec{k}) + |\vec{k}|^2 e^{2ik_0t} \hat{a}^+(\vec{k}) \hat{a}^+(-\vec{k}) \Big]$$

$$③ \int d^3x \; \frac{1}{2} m^2 \varphi^2(x)$$

$$= \frac{1}{2}\int \frac{d^3\vec{k}}{2E(k)} m^2 \Big[e^{-2ik_0t} \hat{a}(\vec{k}) \hat{a}(-\vec{k}) + \hat{a}(\vec{k}) \hat{a}^+(\vec{k}) + \hat{a}^+(\vec{k}) \hat{a}(\vec{k}) + e^{2ik_0t} \hat{a}^+(\vec{k}) \hat{a}^+(-\vec{k}) \Big]$$

$$H = ① + ② + ③$$

$$= \frac{1}{2}\int \frac{d^3\vec{k}}{2E(\vec{k})} \; 2k_0^2 \left[\hat{a}(\vec{k}) \hat{a}^+(\vec{k}) + \hat{a}^+(\vec{k}) \hat{a}(\vec{k}) \right]$$

$$= \frac{1}{2}\int d^3\vec{k} \; E(\vec{k}) \left[\hat{a}(\vec{k}) \hat{a}^+(\vec{k}) + \hat{a}^+(\vec{k}) \hat{a}(\vec{k}) \right].$$

(b) $\vec{P} = -\int d^3\vec{x}\ \hat{\pi}(x)\ \nabla\hat{\varphi}(x)$

$$= -\int d^3\vec{x} \int d^3\vec{k} \int d^3\vec{k}' \left[-ik_0 f_{\vec{k}}\,\hat{a}(\vec{k}) + ik_0 f^*_{\vec{k}}\,\hat{a}^+(\vec{k})\right] \cdot \left[i\vec{k}' f_{\vec{k}'}\,\hat{a}(\vec{k}') - i\vec{k}' f^*_{\vec{k}'}\,\hat{a}^+(\vec{k}')\right]$$

$$= -\int \frac{d^3\vec{k}}{2E(k)} \left[-k_0\vec{k}\, e^{-2ik_0t}\,\hat{a}(\vec{k})\,\hat{a}(-\vec{k}) - k_0\vec{k}\,\hat{a}(\vec{k})\hat{a}^+(\vec{k}) - k_0\vec{k}\,\hat{a}^+(\vec{k})\,\hat{a}(\vec{k}) - k_0\vec{k}\,\hat{a}^+(\vec{k})\,\hat{a}^+(-\vec{k})\,e^{2ik_0t}\right]$$

$$= \frac{1}{2}\int d^3\vec{k}\ \vec{k}\left[+e^{-2ik_0t}\,\hat{a}(\vec{k})\,\hat{a}(-\vec{k}) + \hat{a}(\vec{k})\,\hat{a}^+(\vec{k}) + \hat{a}^+(\vec{k})\,\hat{a}(\vec{k}) + e^{2ik_0t}\,\hat{a}^+(\vec{k})\,\hat{a}^+(-\vec{k})\right]$$

As $[\hat{a}(\vec{k}), \hat{a}(\vec{k}')] = [\hat{a}^+(\vec{k}), \hat{a}^+(\vec{k}')] = 0$, terms like $\vec{k}\,e^{-2ik_0t}\,\hat{a}(\vec{k})\hat{a}(-\vec{k})$ and $\vec{k}\,e^{+2ik_0t}\,\hat{a}^+(\vec{k})\,\hat{a}^+(-\vec{k})$ are odd functions of $\vec{k}$. They don't contribute to the integral.

Finally we have:

$$\vec{P} = \frac{1}{2}\int d^3\vec{k}\ \vec{k}\left[\hat{a}^+(\vec{k})\,\hat{a}(\vec{k}) + \hat{a}(\vec{k})\,\hat{a}^+(\vec{k})\right].$$