Phys 483
HW3

Problem 1. MIT bag model of Quark confinement

Starting from Eq. 10.410: $\psi(x^\mu) = e^{-i\varepsilon t}\begin{pmatrix}\phi(\vec{r})\\ \chi(\vec{r})\end{pmatrix}$

The Dirac equation reads then, according to (10.411, 10.412),

$$\begin{cases} \vec{\sigma}\cdot\hat{\vec{p}}\,\chi + m\phi + V(r)\,\phi = \varepsilon\phi \\ \vec{\sigma}\cdot\hat{\vec{p}}\,\phi - m\chi + V(r)\,\chi = \varepsilon\chi . \end{cases}$$

We will use a slightly different form of (10.420 10.421),

$$\begin{cases} \begin{pmatrix}\phi(\vec{r})\\ \chi(\vec{r})\end{pmatrix} = \begin{pmatrix} i f_1(r)\, \mathcal{Y}_{jm}(j+\frac{1}{2},\frac{1}{2}|\hat{r}) \\ -g_1(r)\,\mathcal{Y}_{jm}(j-\frac{1}{2},\frac{1}{2}|\hat{r})\end{pmatrix} \quad \text{--- (1)} \\ \begin{pmatrix}\phi(\vec{r})\\ \chi(\vec{r})\end{pmatrix} = \begin{pmatrix} i f_2(r)\, \mathcal{Y}_{jm}(j-\frac{1}{2},\frac{1}{2}|\hat{r}) \\ -g_2(r)\,\mathcal{Y}_{jm}(j+\frac{1}{2},\frac{1}{2}|\hat{r})\end{pmatrix} \quad \text{--- (2)} \end{cases}$$

Solution (1) and (2) are also eigenstate of parity operator $\hat{P} = \gamma^0 \hat{\mathcal{P}}$.

(a) $P = \gamma^0 \hat{P}$, $\quad H = \begin{pmatrix} V+m & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & V-m \end{pmatrix}$

$$[\gamma^0 \hat{P}, H] = \left[\begin{pmatrix} \hat{P} & 0 \\ 0 & -\hat{P} \end{pmatrix}\begin{pmatrix} V+m & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & V-m \end{pmatrix} - \begin{pmatrix} V+m & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & V-m \end{pmatrix}\begin{pmatrix} \hat{P} & 0 \\ 0 & -\hat{P} \end{pmatrix}\right]$$

$$= \begin{pmatrix} (V+m)\hat{P} & \hat{P}(\vec{\sigma}\cdot\vec{p}) \\ -\hat{P}(\vec{\sigma}\cdot\vec{p}) & (m-V)\hat{P} \end{pmatrix} - \begin{pmatrix} (V+m)\hat{P} & -(\vec{\sigma}\cdot\vec{p})\hat{P} \\ (\vec{\sigma}\cdot\vec{p})\hat{P} & (m-V)\hat{P} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \{\hat{P}, \vec{\sigma}\cdot\vec{p}\} \\ -\{\hat{P}, \vec{\sigma}\cdot\vec{p}\} & 0 \end{pmatrix}$$

$$\{\hat{P}, \vec{\sigma}\cdot\vec{p}\} = \sum_{i=1}^{3} \left[-i\hat{P}\sigma_i\partial_i - i\sigma_i\partial_i\hat{P}\right]$$

$$= \sum_{i=1}^{3} \left[i\sigma_i\partial_i\hat{P} - i\sigma_i\partial_i\hat{P}\right]$$

$$= 0$$

P commutes with H.

$$P^2 = \gamma^0\hat{P}\gamma^0\hat{P} = 1, \qquad P = \pm 1$$

$$\hat{P}\,Y_{lm}(\hat{r}) = (-1)^l\, Y_{lm}(\hat{r})$$

$$P\begin{pmatrix} i f_1(r)\, \mathcal{Y}_{jm}(j+\tfrac{1}{2}, \tfrac{1}{2}|\hat{r}) \\ -g_1(r)\, \mathcal{Y}_{jm}(j-\tfrac{1}{2}, \tfrac{1}{2}|\hat{r}) \end{pmatrix} = (-1)^{j+\frac{1}{2}} \begin{pmatrix} i f_1(r)\, \mathcal{Y}_{jm}(j+\tfrac{1}{2}, \tfrac{1}{2}|\hat{r}) \\ -g_1(r)\, \mathcal{Y}_{jm}(j-\tfrac{1}{2}, \tfrac{1}{2}|\hat{r}) \end{pmatrix}$$

$$P\begin{pmatrix} i f_2(r)\, \mathcal{Y}_{jm}(j-\tfrac{1}{2}, \tfrac{1}{2}|\hat{r}) \\ -g_2(r)\, \mathcal{Y}_{jm}(j+\tfrac{1}{2}, \tfrac{1}{2}|\hat{r}) \end{pmatrix} = (-1)^{j-\frac{1}{2}} \begin{pmatrix} i f_2(r)\, \mathcal{Y}_{jm}(j-\tfrac{1}{2}, \tfrac{1}{2}|\hat{r}) \\ -g_2(r)\, \mathcal{Y}_{jm}(j+\tfrac{1}{2}, \tfrac{1}{2}|\hat{r}) \end{pmatrix}$$

(b) Recall how $\vec{\sigma}\cdot\hat{\vec{p}}$ acts on the spin-angular momentum states $\mathcal{Y}_{jm}(j\pm\frac{1}{2},\frac{1}{2}|\hat{r})$ (10.413, 10.414)

$$\begin{cases} \vec{\sigma}\cdot\hat{\vec{p}}\, f(r)\, \mathcal{Y}_{jm}(j+\frac{1}{2},\frac{1}{2}|\hat{r}) = i\left[\partial_r + \frac{j+\frac{3}{2}}{r}\right] f(r)\, \mathcal{Y}_{jm}(j-\frac{1}{2},\frac{1}{2}|\hat{r}) \\ \vec{\sigma}\cdot\hat{\vec{p}}\, g(r)\, \mathcal{Y}_{jm}(j-\frac{1}{2},\frac{1}{2}|\hat{r}) = i\left[\partial_r + \frac{\frac{1}{2}-j}{r}\right] g(r)\, \mathcal{Y}_{jm}(j+\frac{1}{2},\frac{1}{2}|\hat{r}) \end{cases}$$

$f(r)$ and $g(r)$ then satisfy:

$$\boxed{\begin{aligned} &\begin{cases} \left(\partial_r + \frac{\frac{1}{2}-j}{r}\right) g_1 - (m+V-E)\, f_1 = 0 \\ \left(\partial_r + \frac{j+\frac{3}{2}}{r}\right) f_1 + (-m+V-E)\, g_1 = 0 \end{cases} \qquad \text{——— (3)} \\ &\text{and} \\ &\begin{cases} \left(\partial_r + \frac{j+\frac{3}{2}}{r}\right) g_2 + (E-m-V)\, f_2 = 0 \\ \left(\partial_r + \frac{-j+\frac{1}{2}}{r}\right) f_2 + (V-m-E)\, g_2 = 0 \end{cases} \qquad \text{——— (4)} \end{aligned}}$$

(c) In MIT bag model, $V=0$. The equations reduce to:

$$\begin{cases} \left(\partial_r + \frac{\frac{1}{2}-j}{r}\right) g_1 + (E-m)\, f_1 = 0 \\ \left(\partial_r + \frac{j+\frac{3}{2}}{r}\right) f_1 - (m+E)\, g_1 = 0 \end{cases} \qquad \text{——— (5)}$$

and

$$\begin{cases} \left(\partial_r + \frac{j+\frac{3}{2}}{r}\right) g_2 + (E-m)\, f_2 = 0 \\ \left(\partial_r + \frac{\frac{1}{2}-j}{r}\right) f_2 - (m+E)\, g_2 = 0 \end{cases} \qquad \text{——— (6)}$$

One can further get the differential equations of f_1 and f_2 by eliminating g_1 and g_2.

· This leads to·

$$\begin{cases} \left[\partial_r^2 + \frac{2}{r}\partial_r - \frac{(j+\frac{1}{2})(j+\frac{3}{2})}{r^2} + (E^2-m^2) \right] f_1 = 0 & \text{—— (7)} \\ \left[\partial_r^2 + \frac{2}{r}\partial_r - \frac{(\frac{1}{2}-j)(-j-\frac{1}{2})}{r^2} + (E^2-m^2) \right] f_2 = 0 & \text{—— (8)} \end{cases}$$

and

$$\begin{cases} g_1 = \frac{1}{m+E}\left(\partial_r + \frac{j+\frac{3}{2}}{r}\right) f_1 & \text{—— (9)} \\ g_2 = \frac{1}{m+E}\left(\partial_r + \frac{\frac{1}{2}-j}{r}\right) f_2 & \text{—— (10)} \end{cases}$$

(d) Recall that the spherical Bessel functions are solutions of the equation

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2} + k_0^2 \right] f_l(k_0 r) = 0 \quad \text{—— (11)}$$

It is benificial then to make the following analogy

$$\begin{cases} \boxed{l = j + \frac{1}{2}} & \text{for Eq (7)} \\ \boxed{l = j - \frac{1}{2}} & \text{for Eq (8).} \end{cases} \quad \text{—— (12)}$$

If $k_0^2 = E^2 - m^2 \geq 0$, the solution of Eq (11) are

$j_l(k_0 r)$ regular as $r \to 0$

$n_l(k_0 r)$ singular as $r \to 0$

while if

$k_0^2 = -K_0^2 = E^2 - m^2 < 0$, the solutions are

$h_\ell^{(1)}(iK_0 r)$ regular as $r \to \infty$

$h_\ell^{(2)}(iK_0 r)$ singular as $r \to \infty$.

For the study of hadrons, we are interested in positive energy solution, $E > 0$. The physically acceptable solutions are:

$$\boxed{j_\ell(k_0 r) \quad \text{if } k_0^2 \geq 0 \quad \text{and} \quad h_\ell^{(1)}(iK_0 r) \text{ if } k_0^2 < 0}$$

Spherical Bessel functions satisfy the recursion relations:

$$\boxed{\begin{cases} \dfrac{2\ell+1}{x} f_\ell(x) = f_{\ell-1}(x) + f_{\ell+1}(x) & \text{—(13)} \\ f'_\ell(x) = \dfrac{1}{2\ell+1}\left[\ell f_{\ell-1}(x) - (\ell+1) f_{\ell+1}(x)\right] & \text{—(14)} \end{cases}}$$

Exploiting the recursion rules (13, 14) to find $g_1(r)$ and $g_2(r)$.

(e) First, let's change variable from r to x, where $x = kr$, so that $k = k_0$ for solution of j and $k = iK_0$ for solution of h.

$$g(kr) = \frac{1}{m+E}\left(\partial_r + \frac{j+\frac{3}{2}}{r}\right) f_1(kr)$$

$$g_1(x) = \frac{k}{m+E}\left(\partial_x + \frac{j+\frac{3}{2}}{x}\right) f_1(x) \qquad \text{— (15)}$$

Combine Eq 13, 14, we have

$$f_\ell'(x) = -\frac{\ell+1}{x} f_\ell(x) + f_{\ell-1}(x) \qquad \text{— (16)}$$

To prove it, let's do it explicitly.

$$f_\ell'(x) = \frac{1}{2\ell+1}\left[\ell f_{\ell-1}(x) - (\ell+1)\left(-f_{\ell-1}(x) + \frac{2\ell+1}{x} f_\ell(x)\right)\right]$$

$$= \frac{1}{2\ell+1}\left[(2\ell+1) f_{\ell-1}(x) - (\ell+1)\frac{2\ell+1}{x} f_\ell(x)\right]$$

$$= f_{\ell-1}(x) - \frac{\ell+1}{x} f_\ell(x)$$

$$g_1(x) = \frac{k}{m+E}\left(f_{\ell-1}(x) - \frac{\ell+1}{x} f_\ell(x) + \frac{j+\frac{3}{2}}{x} f_\ell(x)\right)$$

$$= \frac{k}{m+E}\left[f_{\ell-1}(x) + \left(\frac{\ell+1}{x} - \frac{\ell+1}{x}\right) f_\ell(x)\right]$$

$$\boxed{= \frac{k}{m+E} f_{\ell-1}(x)} \qquad \boxed{\text{for } \ell = j + \tfrac{1}{2}} \qquad \text{— (17)}$$

For the case g_2, f_2, $\ell = j - \frac{1}{2}$. we do it differently.

$$f_\ell'(x) = \frac{1}{2\ell+1}\left[-(\ell+1) f_{\ell+1}(x) + \ell \left(\frac{2\ell+1}{x} f_\ell(x) - f_{\ell+1}(x)\right)\right]$$

$$= \frac{1}{2\ell+1}\left[-(2\ell+1) f_{\ell+1}(x) + \ell \frac{2\ell+1}{x} f_\ell(x)\right]$$

$$= \frac{\ell}{x} f_\ell(x) - f_{\ell+1}(x) \qquad (18)$$

$$g_2(x) = \frac{k}{m+E}\left(\partial_x + \frac{\frac{1}{2} - j}{x}\right) f_2$$

$$= \frac{k}{m+E}\left(\cancel{\frac{\ell}{x}} f_\ell(x) - f_{\ell+1}(x) + \frac{-\ell}{x} f_\ell(x)\right)$$

$$\boxed{= -\frac{k}{m+E} f_{\ell+1}(x)} \quad \boxed{\text{for } \ell = j - \tfrac{1}{2}} \qquad (19)$$

(f) Recall Eq. 6.186,

$$\vec{\sigma} \cdot \hat{r}\, \mathcal{Y}_{jm}(j \pm \tfrac{1}{2}, \tfrac{1}{2} \mid \hat{r}) = -\mathcal{Y}_{jm}(j \mp \tfrac{1}{2}, \tfrac{1}{2} \mid \hat{r}),$$

we can now write our solution even concisely.

① $\ell = j + \frac{1}{2}$

$$\psi^{\ell}_{jm}(\vec{r}) = \begin{pmatrix} i f_\ell(x) \\ \frac{k}{m+E}\, \vec{\sigma}\cdot\hat{r}\, f_{\ell-1}(x) \end{pmatrix} \mathcal{Y}_{jm}(\ell, \tfrac{1}{2} | \hat{r})$$

② $\ell = j - \frac{1}{2}$

$$\psi^{\ell}_{jm}(\vec{r}) = \begin{pmatrix} i f_\ell(x) \\ -\frac{k}{m+E}\, \vec{\sigma}\cdot\hat{r}\, f_{\ell+1}(x) \end{pmatrix} \mathcal{Y}_{jm}(\ell, \tfrac{1}{2} | \hat{r})$$

——— (20)

(g) Now let's go back to the hadron problem.
We look the ground state with $\ell = 0$, $j = \frac{1}{2}$.
It is a positive parity state. We should use the second set of solutions.

① Inside the hadron, $r \le R$, $m = 0$.

$k = k_0 = E$, $\frac{k}{m+E} = 1$

$$\psi_I(r) = N_I \begin{pmatrix} i j_0(Er) \\ -\vec{\sigma}\cdot\hat{r}\, j_1(Er) \end{pmatrix} \mathcal{Y}_{\frac{1}{2}m}(0, \tfrac{1}{2} | \hat{r}),$$

——— (21)

where N_I is the normalization constant.

$$j_0(x) = \frac{\sin x}{x}, \quad j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}$$

② Outside the hadron, $r > R$, $m \to \infty$, $E \ll m$.

$$k = iK_0 = im, \quad \frac{k}{m+E} = i$$

$$\psi_{II}(r) = N_{II} \begin{pmatrix} i h_0^{(1)}(imr) \\ -i\vec{\sigma}\cdot\hat{r}\, h_1^{(1)}(imr) \end{pmatrix} \mathcal{Y}_{\frac{1}{2},m}(0,\tfrac{1}{2}|\hat{r}) \qquad \text{—— (22)}$$

$$h_0^{(1)}(ix) = -\frac{e^{-x}}{x}, \quad h_1^{(1)}(ix) = i\,\frac{e^{-x}}{x}\left(1+\frac{1}{x}\right)$$

(h) We want to take the limit $m \to \infty$, but $\psi_{II}(R)$ doesn't vanish so that the solutions can be connected at $r=R$. This can be solved by choose $N_{II} = -N_{II}' m e^{mR}$

At $r=R$, the wave function is continuous:

$$\begin{cases} \psi_I(R) = N_I \begin{pmatrix} i j_0(ER) \\ -\vec{\sigma}\cdot\hat{r}\, j_1(ER) \end{pmatrix} \mathcal{Y}_{\frac{1}{2}m}(0,\tfrac{1}{2}|\hat{r}) \\ \psi_{II}(R) = N_{II}' \begin{pmatrix} i\frac{1}{R} \\ -\vec{\sigma}\cdot\hat{r}\,\frac{1}{R} \end{pmatrix} \mathcal{Y}_{\frac{1}{2}m}(0,\tfrac{1}{2}\hat{r}) \end{cases} \qquad \text{—— (23)}$$

To satisfy $\psi_I(R) = \psi_{II}(R)$, we obtain

$$j_0(ER) = j_1(ER) \Rightarrow x = 2.043 \text{ and } E_0 = \frac{2.043}{R}.$$

This is the ground state.

(ii) First excited state: $j = \frac{3}{2}$, $l = 1$, $l = j - \frac{1}{2}$, $P\psi'_{\frac{3}{2},m}(\hat{r}) = -\psi'_{\frac{3}{2}m}(\hat{r})$

$$\begin{cases} \psi_I(\vec{r}) = N_I \begin{pmatrix} i j_1(x) \\ -\vec{\sigma}\cdot\hat{r}\, j_2(x) \end{pmatrix} \mathcal{Y}_{\frac{3}{2}m}(1, \tfrac{1}{2}|\hat{r}) \\ \psi_{II}(\vec{r}) = \cancel{i} N'_{II}\, m\, e^{mR} \begin{pmatrix} i h_1^{(1)}(x) \\ -i\, \vec{\sigma}\cdot\hat{r}\, h_2^{(1)}(x) \end{pmatrix} \mathcal{Y}_{\frac{3}{2}m}(1, \tfrac{1}{2}|\hat{r}) \end{cases} \quad \text{—— (24)}$$

Use recursion relation to find $h_2^{(1)}(x)$ and $j_2(x)$

Recursion relation: $f_{l+1} = \frac{2l+1}{x} f_l - f_{l-1}$

$$j_2 = \frac{3}{x} j_1 - j_0, \qquad h_2^{(1)} = \frac{3}{x} h_1^{(1)} - h_0^{(1)}. \quad \text{—— (25)}$$

$$\begin{cases} \lim\limits_{x\to\infty} h_2^{(1)}(x) = -h_0^{(1)} \\ \lim\limits_{x\to\infty} h_1^{(1)}(x) = -i\, h_0^{(1)} \end{cases} \quad \text{—— (26)}$$

At the connecting point $r = R$.

$$\begin{cases} \psi_I(R) = N_I \begin{pmatrix} i\, j_1(ER) \\ -\vec{\sigma}\cdot\hat{r}\, j_2(ER) \end{pmatrix} \mathcal{Y}_{\frac{3}{2}m}(1, \tfrac{1}{2} | \hat{r}) \\ \psi_{II}(R) = N_{II}' m e^{mR} \begin{pmatrix} -i\, h_0^{(1)}(mR) \\ \vec{\sigma}\cdot\hat{r}\, h_0^{(1)}(mR) \end{pmatrix} \mathcal{Y}_{\frac{3}{2}m}(1, \tfrac{1}{2} | \hat{r}) \end{cases} \quad \text{—— (27)}$$

Then $j_1(ER) = j_2(ER)$ —— (28)

$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x},$$

$$j_2(x) = \frac{3}{x} j_1 - j_0 = \frac{3\sin x}{x^3} - \frac{3\cos x}{x^2} - \frac{\sin x}{x} \quad \text{—— (29)}$$

$\Rightarrow E_1 R = 3.204$ and $E_1 = \dfrac{3.204}{R}$

■ Problem 1 (j):

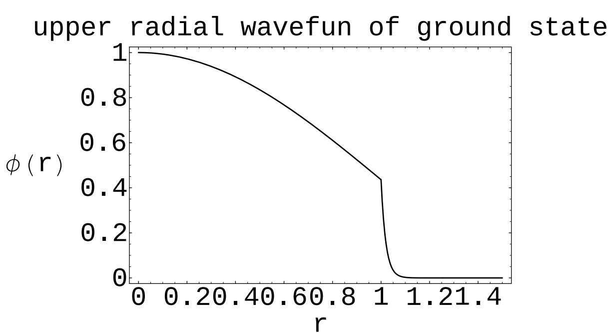
```
In[1]:= << Graphics`Graphics`
        << Graphics`Legend`

        (* Define mass and radius *)
        m = 50; R = 1;
```

■ Ground state

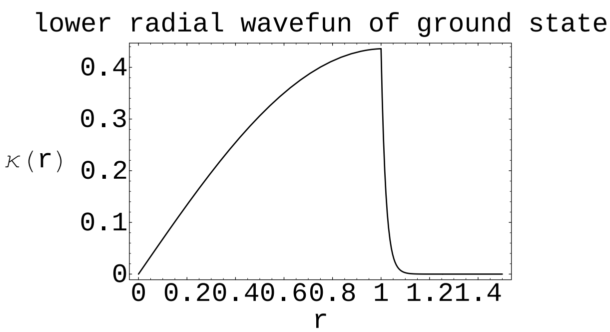
■ 1. upper radial wavefunction

```
e = 2.043;
φg1[x_] := Sin[e x]/(e x); (* inside the sphere *)
φg2[x_] := Exp[-m x]/(m x); (* outside the sphere *)
NN = φg1[1] / φg2[1];    (* scaling factor *)
DisplayTogether[
  Plot[φg1[x], {x, 0, 1}, PlotLabel -> "upper radial wavefun of ground state",
   Frame → True, FrameLabel → {"r", "φ(r)"},
   RotateLabel → False, Axes → False], Plot[φg2[x] * NN, {x, 1, 1.5}]];
```



■ 2. lower radial wavefunction

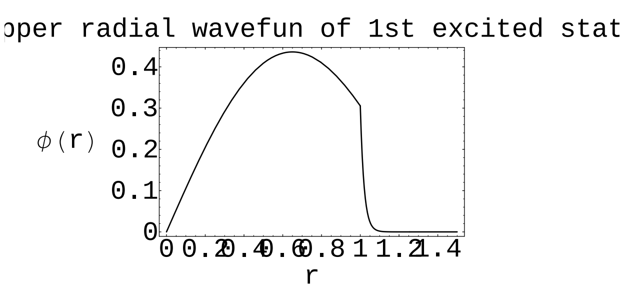
```
κg1[x_] := Sin[e x]/(e x)^2 - Cos[e x]/(e x); (* inside the sphere *)
κg2[x_] := Exp[-m x]/(m x) (1 + 1/(m x)); (* outside the sphere *)
NN = κg1[1] / κg2[1];    (* scaling factor *)
DisplayTogether[
  Plot[κg1[x], {x, 0, 1}, PlotLabel -> "lower radial wavefun of ground state",
   Frame → True, FrameLabel → {"r", "κ(r)"},
   RotateLabel → False, Axes → False], Plot[κg2[x] * NN, {x, 1, 1.5}]];
```



■ the first excited state

■ 1. upper radial wavefunction

```
e = 3.204;
ϕe1[x_] := Sin[e x]/(e x)^2 - Cos[e x]/(e x); (* inside the sphere *)
ϕe2[x_] := Exp[-m x]/(m x) (1 + 1/(m x)) - Exp[-m x]/(m x); (* outside the sphere *)
NN = ϕe1[1] / ϕe2[1];    (* scaling factor *)
DisplayTogether[
  Plot[ϕe1[x], {x, 0, 1}, PlotLabel -> "upper radial wavefun of 1st excited state",
   Frame → True, FrameLabel → {"r", "ϕ(r)"},
   RotateLabel → False, Axes → False], Plot[ϕe2[x] * NN, {x, 1, 1.5}]];
```

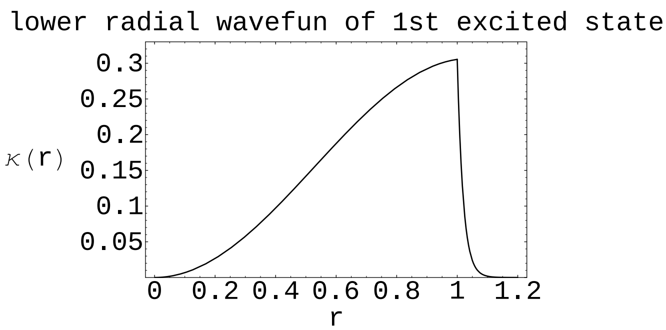


■ 2. lower radial wavefunction

$$\kappa e1[x\_] := \frac{3}{e\,x}\left(\frac{\text{Sin}[e\,x]}{(e\,x)^2} - \frac{\text{Cos}[e\,x]}{e\,x}\right) - \frac{\text{Sin}[e\,x]}{e\,x};\ \text{(* inside the sphere *)}$$

$$\kappa e2[x\_] := \frac{3}{m\,x}\left(\frac{\text{Exp}[-m\,x]}{m\,x}\left(1 + \frac{1}{m\,x}\right)\right) + \frac{\text{Exp}[-m\,x]}{m\,x};\ \text{(* outside the sphere *)}$$

```
NN = κe1[1] / κe2[1];    (* scaling factor *)
DisplayTogether[
  Plot[κe1[x], {x, 0, 1}, PlotLabel -> "lower radial wavefun of 1st excited state",
   Frame → True, FrameLabel → {"r", "κ(r)"}, Axes → False, PlotRange → {0, 0.33},
   RotateLabel → False, Axes → False], Plot[κe2[x] * NN, {x, 1, 1.2}]];
```



■ Problem 2: Relativistic Hydrogen−typeAtom

In[3]:= $\alpha = \dfrac{1}{137.036};$

$m = 3.862 * 10^{-3};$

$np[n\_, j\_] := n - j - \dfrac{1}{2};$

$\gamma[j\_] := \sqrt{\left(j + \dfrac{1}{2}\right)^2 - \alpha^2};$

$\epsilon[n\_, j\_] := \dfrac{1}{\sqrt{1 + \left(\frac{\alpha}{np[n, j] + \gamma[j]}\right)^2}};$

$\mu[n\_, j\_] := \sqrt{(1 - \epsilon[n, j])\,(1 + \epsilon[n, j])};$

■ a. radial wave func for 2 $p_{1/2}$, n=2, j=1/2 and n'=1. Here l=j+1/2, we should use Eq. (10.465−10.468).

In[9]:= $n0 = 2;\ j0 = \dfrac{1}{2};$

In[10]:= `{np[n0, j0], γ[j0], ϵ[n0, j0], μ[n0, j0]}`

Out[10]= `{1, 0.999973, 0.999993, 0.0036487}`

In[11]:= $\kappa 1[j\_] := j + \frac{1}{2};$

$fa[n\_, j\_, r\_] := 1 - \frac{np[n, j]}{2\gamma[j]+1} 2\mu[n, j]\, r;$

$fb[n\_, j\_, r\_] := 1 + \frac{1 - np[n, j]}{2\gamma[j]+1} 2\mu[n, j]\, r;$

$F1[n\_, j\_, r\_] := (2\mu[n, j]\, r)^{\gamma[j]-1} Exp[-\mu[n, j]\, r] \left(\left(\frac{np[n, j] + \gamma[j]}{\epsilon[n, j]} - \kappa 1[j]\right) fa[n, j, r] - np[n, j]\, fb[n, j, r]\right);$

$G1[n\_, j\_, r\_] := (2\mu[n, j]\, r)^{\gamma[j]-1} Exp[-\mu[n, j]\, r] \left(\left(\frac{np[n, j] + \gamma[j]}{\epsilon[n, j]} - \kappa 1[j]\right) fa[n, j, r] + np[n, j]\, fb[n, j, r]\right);$

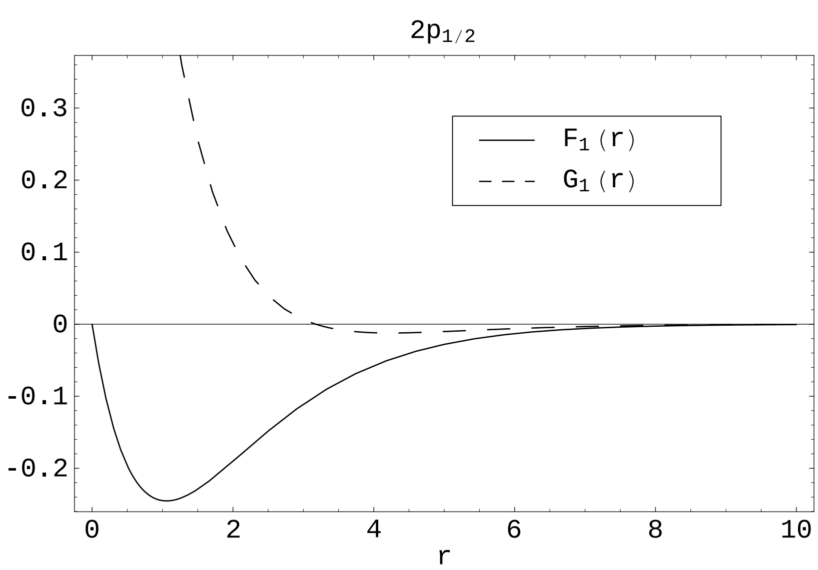
In[20]:= $N1[n\_, j\_] := \frac{(2\mu[n, j])^{1.5}}{Gamma[2\gamma[j]+1]} \sqrt{\frac{(1 + \epsilon[n, j])\, Gamma[2\gamma[j] + np[n, j] + 1]}{4 \frac{np[n,j]+\gamma[j]}{\epsilon[n,j]} \left(\frac{np[n,j]+\gamma[j]}{\epsilon[n,j]} - \kappa 1[j]\right)}};$

$N2[n\_, j\_] := \frac{(2\mu[n, j])^{1.5}}{Gamma[2\gamma[j]+1]} \sqrt{\frac{(1 - \epsilon[n, j])\, Gamma[2\gamma[j] + np[n, j] + 1]}{4 \frac{np[n,j]+\gamma[j]}{\epsilon[n,j]} \left(\frac{np[n,j]+\gamma[j]}{\epsilon[n,j]} - \kappa 1[j]\right)}};$

```
{N1[n0, j0], N2[n0, j0]} (* N1>>N2 *)
```

Out[22]= $\{0.000381749, 6.96447 \times 10^{-7}\}$

In[24]:=
```
Plot[{F1[n0, j0, r / m], G1[n0, j0, r / m]},
  {r, 0, 10}, Frame → True, FrameLabel → {"r", ""},
  RotateLabel → False, PlotStyle → {GrayLevel[0], Dashing[{.03}]}, PlotLabel -> "2p1/2",
  PlotLegend → {"F1(r)", "G1(r)"}, LegendPosition -> {0.1, 0.2},
  LegendSize -> {0.6, 0.2}, LegendShadow → {0, 0}];
```



■ b. radial wave func for 2 $p_{3/2}$, n=2, j=3/2 and n'=0. Here l=j−1/2,we should use Eq. (10.471-10.472).

In[25]:= $n0 = 2;\ j0 = \frac{3}{2};$

In[26]:= {np[n0, j0], γ[j0], ϵ[n0, j0], μ[n0, j0]}

Out[26]= {0, 1.99999, 0.999993, 0.00364868}

In[30]:= $\kappa2[j\_] := -j - \frac{1}{2};$

$F2[n\_, j\_, r\_] := (2\,\mu[n, j]\,r)^{\gamma[j]-1}\,\mathrm{Exp}[-\mu[n, j]\,r]\left(\left(\frac{np[n, j] + \gamma[j]}{\epsilon[n, j]} - \kappa2[j]\right) fa[n, j, r] - np[n, j]\,fb[n, j, r]\right);$

$G2[n\_, j\_, r\_] := (2\,\mu[n, j]\,r)^{\gamma[j]-1}\,\mathrm{Exp}[-\mu[n, j]\,r]\left(\left(\frac{np[n, j] + \gamma[j]}{\epsilon[n, j]} - \kappa2[j]\right) fa[n, j, r] + np[n, j]\,fb[n, j, r]\right);$

In[27]:= $N1[n\_, j\_] := \frac{(2\,\mu[n, j])^{1.5}}{\mathrm{Gamma}[2\,\gamma[j] + 1]}\sqrt{\frac{(1 + \epsilon[n, j])\,\mathrm{Gamma}[2\,\gamma[j] + np[n, j] + 1]}{4\,\frac{np[n,j]+\gamma[j]}{\epsilon[n,j]}\left(\frac{np[n,j]+\gamma[j]}{\epsilon[n,j]} - \kappa2[j]\right)}};$

$N2[n\_, j\_] := \frac{(2\,\mu[n, j])^{1.5}}{\mathrm{Gamma}[2\,\gamma[j] + 1]}\sqrt{\frac{(1 - \epsilon[n, j])\,\mathrm{Gamma}[2\,\gamma[j] + np[n, j] + 1]}{4\,\frac{np[n,j]+\gamma[j]}{\epsilon[n,j]}\left(\frac{np[n,j]+\gamma[j]}{\epsilon[n,j]} - \kappa2[j]\right)}};$

In[33]:= {N1[n0, j0], N2[n0, j0]} (* N1>>N2 *)

Out[33]= {0.000031812, 5.8036×10^{-8}}

```
In[39]:= Plot[{F2[n0, j0, r / m], G2[n0, j0, r / m]},
           {r, 0, 10}, Frame → True, FrameLabel → {"r", ""},
           RotateLabel → False, PlotStyle → {GrayLevel[0], Dashing[{.03}]}, PlotLabel -> "2p3/2",
           PlotLegend → {"F2 (r)", "G2 (r)"}, LegendPosition -> {0.15, 0.1},
           LegendSize -> {0.6, 0.2}, LegendShadow → {0, 0}];
```