2. Show that the wave function for a free spin-$\frac{1}{2}$ particle moving in arbitrary direction is

$$\psi(\vec{p}\lambda,\sigma|\vec{r}t)=\sqrt{\frac{E_p+m}{2m}}\,e^{i(\vec{p}\cdot\vec{r}-\lambda E_p t)}\begin{cases} w_1(\vec{p}), & \lambda=+,\ \sigma=\frac{1}{2} \\ w_2(\vec{p}), & \lambda=+,\ \sigma=-\frac{1}{2} \\ w_3(\vec{p}), & \lambda=-,\ \sigma=\frac{1}{2} \\ w_4(\vec{p}), & \lambda=-,\ \sigma=-\frac{1}{2} \end{cases}$$

where $E_p=\sqrt{m^2+p^2}$ and

$$w_1(\vec{p})=\begin{pmatrix} 1 \\ 0 \\ \frac{p_3}{E_p+m} \\ \frac{p_+}{E_p+m} \end{pmatrix}; \qquad w_2(\vec{p})=\begin{pmatrix} 0 \\ 1 \\ \frac{p_-}{E_p+m} \\ \frac{-p_3}{E_p+m} \end{pmatrix};$$

$$w_3(\vec{p})=\begin{pmatrix} \frac{-p_3}{E_p+m} \\ \frac{-p_+}{E_p+m} \\ 1 \\ 0 \end{pmatrix}; \qquad w_4(\vec{p})=\begin{pmatrix} \frac{-p_-}{E_p+m} \\ \frac{p_3}{E_p+m} \\ 0 \\ 1 \end{pmatrix};$$

where $p_\pm = p_1 \pm i p_2$.

The general solution of the wave function for a free spin-$\frac{1}{2}$ particle are Eq 10.286 and 10.289,

$$\begin{cases} \psi(\vec{p}+1|x^\mu) = \sqrt{\frac{E_p+m}{2m}} \begin{pmatrix} u \\ \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m} u \end{pmatrix} e^{i(\vec{p}\cdot\vec{r}-E_p t)} ; \\ \psi(\vec{p}-1|x^\mu) = \sqrt{\frac{E_p+m}{2m}} \begin{pmatrix} -\frac{\vec{\sigma}\cdot\vec{p}}{E_p+m} u \\ u \end{pmatrix} e^{i(\vec{p}\cdot\vec{r}+E_p t)} . \end{cases}$$

$$\vec{\sigma}\cdot\vec{p} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} p_1 + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} p_2 + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} p_3$$

$$= \begin{pmatrix} p_3 & p_- \\ p_+ & -p_3 \end{pmatrix}$$

We choose the basis as: $u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $u_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Otherwise we have a rotational transformation, $\exp(-\frac{i}{2}\vec{\theta}(\vec{p})\cdot\vec{\sigma})$, which can be regarded as an overall factor. Stick to the simplest basis,

$$\vec{\sigma}\cdot\vec{p}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} p_3 \\ p_+ \end{pmatrix}, \qquad \vec{\sigma}\cdot\vec{p}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} p_- \\ -p_3 \end{pmatrix} .$$

$$\psi(\vec{p}\lambda,\sigma|\vec{r},t)=\sqrt{\frac{E_p+m}{2m}}\,e^{i(\vec{p}\cdot\vec{r}-\lambda E_p t)}\;W_i(\vec{p}),$$

where $i=1,2,3,4$.

$$W_1(\vec{p})=W_{+,\frac{1}{2}}(\vec{p})=\begin{pmatrix}1\\0\\\frac{p_3}{E_p+m}\\\frac{p_+}{E_p+m}\end{pmatrix},\quad W_2(\vec{p})=W_{+,-\frac{1}{2}}(\vec{p})=\begin{pmatrix}0\\1\\\frac{p_-}{E_p+m}\\\frac{-p_3}{E_p+m}\end{pmatrix};$$

$$W_3(\vec{p})=W_{-\frac{1}{2}}(\vec{p})=\begin{pmatrix}\frac{-p_3}{E_p+m}\\\frac{-p_+}{E_p+m}\\1\\0\end{pmatrix};\quad W_4(\vec{p})=W_{-,-\frac{1}{2}}(\vec{p})=\begin{pmatrix}\frac{-p_-}{E_p+m}\\\frac{p_3}{E_p+m}\\0\\1\end{pmatrix}.$$

It can be proved that $W_i^+ W_j=0$ for $i\neq j$.

3. The initial wave function of a spin-$\frac{1}{2}$ particle is $\psi(\vec{r}, t=0) = \left(\frac{1}{\pi d^2}\right)^{\frac{3}{4}} e^{-r^2/2d^2} \,w_1(\vec{p}=0)$.

(a) First we expand $\psi(\vec{r}, t=0)$ into momentum space

$$\int \frac{d^3\vec{r}}{(2\pi)^3}\, \psi(\vec{r}, t=0)\, e^{i\vec{p}\cdot\vec{r}}$$

$$= \int \frac{d^3\vec{r}}{(2\pi)^3} \left(\frac{1}{\pi d^2}\right)^{\frac{3}{4}} e^{-r^2/2d^2}\, \psi_1(0)\, e^{i\vec{p}\cdot\vec{r}}$$

$$= \left(\frac{d}{2\pi}\right)^{\frac{3}{2}} \left(\frac{1}{\pi}\right)^{\frac{3}{4}} e^{-\frac{p^2 d^2}{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= A\, W_0 \,,$$

where $A \equiv \left(\frac{d}{2\pi}\right)^{\frac{3}{2}} \left(\frac{1}{\pi}\right)^{\frac{3}{4}} e^{-\frac{p^2 d^2}{2}}$, $\quad W_0 \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

The time dependant wave function, in general, is

$$\psi(\vec{r}, t) = \int d^3p \sum_{j=1}^{4} b_j(\vec{p})\, w_j(\vec{p})\, e^{i(\vec{p}\cdot\vec{r} - \lambda_j E_p t)},$$

$\lambda_{1,2} = 1, \quad \lambda_{3,4} = -1.$

Exploiting the orthogonal property of $w_j(\vec{p})$, we can then work out the coefficients $b_j(\vec{p})$.

$$A w_0 = \sum_j b_j w_j(\vec{p})$$

Take the inner product between w_0 and w_j

$$A w_j^{+} w_0 = \sum_{j'} b_{j'} w_j^{+} w_{j'}$$

$$= \sum_j b_{j'} |w_j|^2 \delta_{jj'}$$

$$= b_j |w_j|^2$$

$$\boxed{b_j = \frac{A w_j^{+} w_0}{|w_j|^2}}$$

Let's examine $|w_j|^2$ now.

$$|w_1(\vec{p})|^2 = 1 + \frac{p_3^2}{(E+m)^2} + \frac{p_1^2+p_2^2}{(E+m)^2} = \frac{(E+m)^2+p^2}{(E+m)^2}$$

$$(E+m)^2+p^2 = (E+m)^2 + (E^2-m^2) = E^2 + (E+m+m)(E+m-m)$$

$$= E^2 + E(E+2m) = 2E(E+m)$$

$$\boxed{|w_1(\vec{p})|^2 = \frac{2E(E+m)}{(E+m)^2} = \frac{2E}{E+m}}$$

Similarly: $|w_2(\vec{p})|^2 = |w_3(\vec{p})|^2 = |w_4(\vec{p})|^2 = \frac{2E}{E+m}$

$$b_1 = A\ w_1^+ w_0 \frac{E+m}{2E} = A\frac{E+m}{2E}$$

$$b_2 = A\ w_2^+ w_0 \frac{E+m}{2E} = A \cdot 0 = 0$$

$$b_3 = A\ w_3^+ w_0 \frac{E+m}{2E} = A\left(-\frac{p_3}{E+m}\right)\left(\frac{E+m}{2E}\right) = -A\frac{p_3}{2E}$$

$$b_4 = A\ w_4^+ w_0 \frac{E+m}{2E} = A\left(-\frac{p_+}{E+m}\right)\left(\frac{E+m}{2E}\right) = -A\frac{p_+}{2E}$$

(b) $N(\vec{p}) = |b_1(\vec{p})|^2 + |b_2(\vec{p})|^2 + |b_3(\vec{p})|^2 + |b_4(\vec{p})|^2$

$$= A^2\left[\left(\frac{E+m}{2E}\right)^2 + \frac{p_3^2}{(2E)^2} + \frac{p_1^2+p_2^2}{(2E)^2}\right]$$

$$= A^2\ \frac{(E+m)^2+(E^2-m^2)}{4E^2}$$

$$= A^2\ \frac{E+m}{2E}$$

$$R_+ = \frac{|b_1(\vec{p})|^2 + |b_2(\vec{p})|^2}{N(\vec{p})} = \left(\frac{E+m}{2E}\right)^2 \cdot \frac{2E}{E+m} = \frac{E+m}{2E}$$

$$R_- = \frac{|b_3(\vec{p})|^2 + |b_4(\vec{p})|^2}{N(\vec{p})} = 1 - R_+ = \frac{E-m}{2E}$$

The point of this problem is to see how the weight of particle and anti-particle changes due to the width of wave packet d. The Fourier of a Gaussian wave packet $e^{-r^2/2d^2}$ is also a wave packet $e^{-p^2d^2/2}$. The flatter the packet in coordinate space, the narrower it is in the momentum space. This is governed by the uncertainty principle. To simplify our discussion, we are going to look at part of the packet in the momentum space, say $|p| \leq \frac{\sqrt{2}}{d}$, where the Gaussian drops to e^{-1}. We study three cases: ① $m=1$, $d=10$; ② $m=1$, $d=1$, and ③ $m=1$, $d=0.1$. As we can see next, the contribution of anti-particle becomes more and more pronounced as d gets smaller.

■ Problem 3 (b):

```
<< Graphics`Graphics`
<< Graphics`Legend`
```

$$\text{EE}[p\_, m\_] := \sqrt{p^2 + m^2};$$

$$\text{RPlus}[p\_, d\_, m\_] := \frac{\text{EE}[p, m] + m}{2\,\text{EE}[p, m]};$$

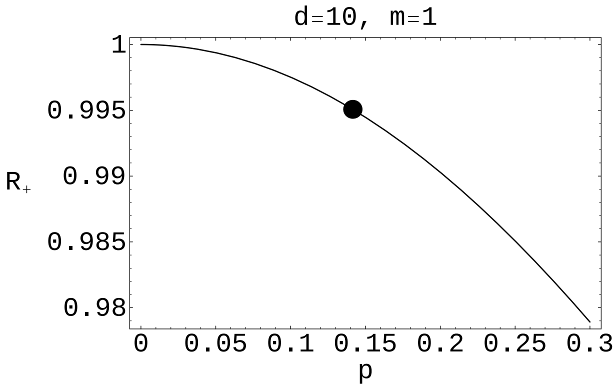
$$\text{RMinus}[p\_, d\_, m\_] := 1 - \text{RPlus}[p, d, m];$$

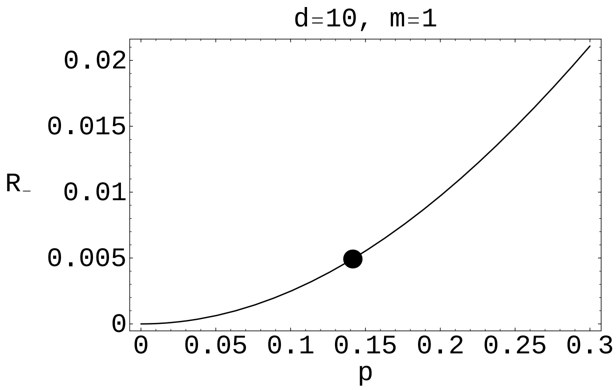
$$\sigma[d\_] := \sqrt{2.0} \Big/ d;$$

```
{σ[10], σ[1], σ[0.1]}
```

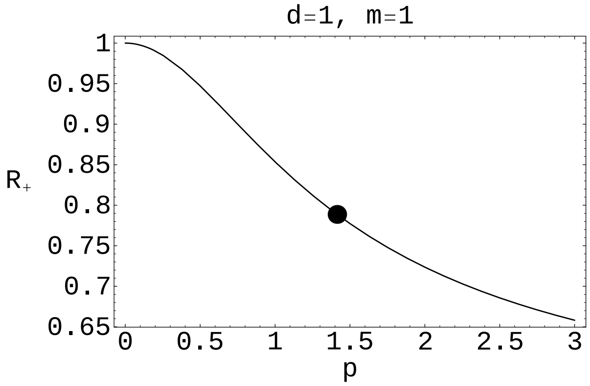
```
{0.141421, 1.41421, 14.1421}
```

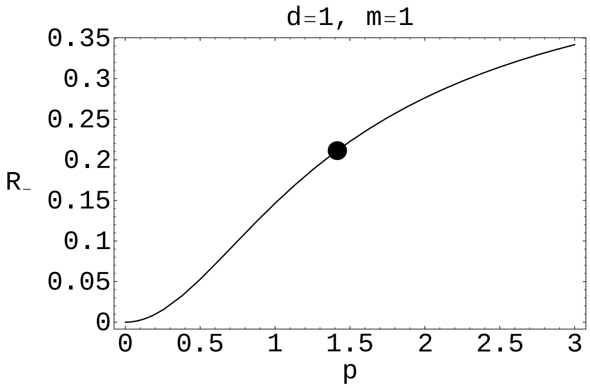
```
DisplayTogether[Plot[RPlus[p, 10, 1], {p, 0, .3},
  PlotLabel → "d=10, m=1", Frame → True, FrameLabel → {"p", "R+"},
  RotateLabel → False], ListPlot[{{σ[10], RPlus[σ[10], 10, 1]}},
  PlotStyle -> PointSize[0.04]]]; DisplayTogether[Plot[RMinus[p, 10, 1],
  {p, 0, .3}, PlotLabel → "d=10, m=1", Frame → True, FrameLabel → {"p", "R-"},
  RotateLabel → False], ListPlot[{{σ[10], RMinus[σ[10], 10, 1]}},
  PlotStyle -> PointSize[0.04]]];
```



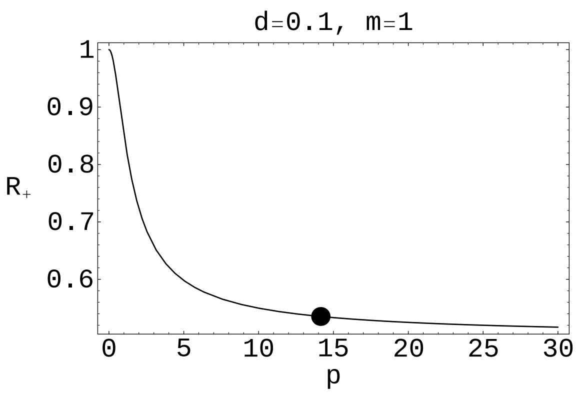


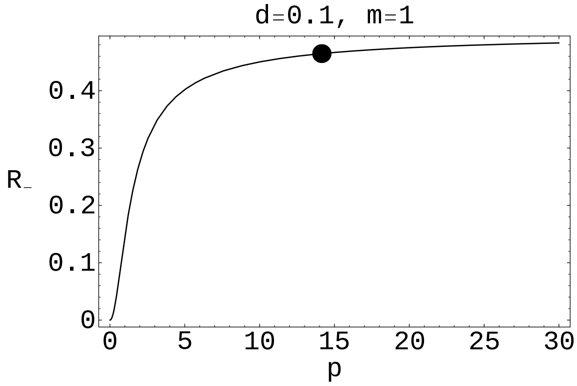
```
DisplayTogether[Plot[RPlus[p, 1, 1], {p, 0, 3},
   PlotLabel → "d=1, m=1", Frame → True, FrameLabel → {"p", "R₊"},
   RotateLabel → False, Axes → False],
  ListPlot[{{σ[1], RPlus[σ[1], 1, 1]}}, PlotStyle -> PointSize[0.04]]];
DisplayTogether[Plot[RMinus[p, 1, 1], {p, 0, 3},
   PlotLabel → "d=1, m=1", Frame → True, FrameLabel → {"p", "R₋"},
   RotateLabel → False], ListPlot[{{σ[1], RMinus[σ[1], 1, 1]}},
   PlotStyle -> PointSize[0.04]]];
```






```
DisplayTogether[Plot[RPlus[p, .1, 1], {p, 0, 30},
   PlotLabel → "d=0.1, m=1", Frame → True, FrameLabel → {"p", "R₊"},
   RotateLabel → False], ListPlot[{{σ[.1], RPlus[σ[.1], .1, 1]}},
   PlotStyle -> PointSize[0.04]]];
DisplayTogether[Plot[RMinus[p, .1, 1], {p, 0, 30}, PlotLabel → "d=0.1, m=1",
   Frame → True, FrameLabel → {"p", "R₋"},
   RotateLabel → False], ListPlot[{{σ[.1], RMinus[σ[.1], .1, 1]}},
   PlotStyle -> PointSize[0.04]]];
```




d=0.1, m=1
0.4
0.3
R_
0.2
0.1
0
0
5
10
15
20
25
30
p

(c) The charge density is defined as $n = \psi^+ \gamma^0 \psi$.

Then ~~the~~ charge density in each state is

$$n_j = |b_j(\vec{p})|^2 \, w_j(\vec{p})^+ \gamma^0 w_j(\vec{p})$$

$$w_1^+(\vec{p})\,\gamma^0\, w_1(\vec{p}) = 1 - \frac{p_3^2 + p_1^2 + p_2^2}{(E+m)^2} = 1 - \frac{p^2}{(E+m)^2} = \frac{2m}{E+m}$$

$$w_2^+(\vec{p})\,\gamma^0\, w_2(\vec{p}) = 1 - \frac{p_1^2 + p_2^2 + p_3^2}{(E+m)^2} = \frac{2m}{E+m}$$

$$w_3^+(\vec{p})\,\gamma^0\, w_3(\vec{p}) = \frac{p_1^2 + p_2^2 + p_3^2}{(E+m)^2} - 1 = -\frac{2m}{E+m}$$

$$w_4^+(\vec{p})\,\gamma^0\, w_4(\vec{p}) = -\frac{2m}{E+m}$$

$$n_1 = A^2 \cdot \left(\frac{E+m}{2E}\right)^2 \cdot \frac{2m}{E+m} = A^2 \frac{(E+m)m}{2E^2}$$

$$n_2 = 0$$

$$n_3 = A^2 \left(\frac{p_3}{2E}\right)^2 \left(-\frac{2m}{E+m}\right) = -A^2 \frac{p_3^2\, m}{2E^2(E+m)}$$

$$n_4 = A^2 \frac{p_1^2 + p_2^2}{(2E)^2} \cdot \left(-\frac{2m}{E+m}\right) = -A^2 \frac{(p_1^2 + p_2^2)\, m}{2E^2(E+m)}$$

(d) If we start with a non-relativistic spin-$\frac{1}{2}$ particle, $\Psi(\vec{r}, t=0) = \left(\frac{1}{\pi d^2}\right)^{\frac{3}{4}} e^{-r^2/2d^2} |\frac{1}{2}, +\frac{1}{2}\rangle$.

It can only evolve as a spin-$\frac{1}{2}$ particle if there is not other external perturbation.

$$\Psi(\vec{r}, t) = \int d^3\vec{p} \; e^{i(\vec{p}\cdot\vec{r} - E_p t)} \left(b_+(\vec{p}) |\tfrac{1}{2}, +\tfrac{1}{2}\rangle + b_-(\vec{p}) |\tfrac{1}{2}, -\tfrac{1}{2}\rangle \right).$$

The coefficients are:

$$\begin{cases} b_+(\vec{p}) = \left(\frac{d}{2\pi}\right)^{\frac{3}{2}} \left(\frac{1}{\pi}\right)^{\frac{3}{4}} e^{-\frac{p^2 d^2}{2}} = A \\ b_-(\vec{p}) = 0 \end{cases}$$

This implies a wave packet of non-relativistic particle in coordinat space can be constructed from the corresponding spin state. However, to construct such a wave-packet of non-relativistic particle, both particle and anti-particle states are required. Particularly, for the anti-particle, both spin $\frac{1}{2}$ and $-\frac{1}{2}$ states are required.