Problem Set I

phys 483 / Fall 2002

2 (a) Prove Eq. 10.49 for $k=2$, $l=3$.

① $[J_2, J_3] = J_1$

$$\begin{pmatrix} 0&0&0&0\\ 0&0&0&1\\ 0&0&0&0\\ 0&-1&0&0 \end{pmatrix}\begin{pmatrix} 0&0&0&0\\ 0&0&-1&0\\ 0&1&0&0\\ 0&0&0&0 \end{pmatrix} - \begin{pmatrix} 0&0&0&0\\ 0&0&-1&0\\ 0&1&0&0\\ 0&0&0&0 \end{pmatrix}\begin{pmatrix} 0&0&0&0\\ 0&0&0&1\\ 0&0&0&0\\ 0&-1&0&0 \end{pmatrix}$$

$$= \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&1&0 \end{pmatrix} - \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&1\\ 0&0&0&0 \end{pmatrix} = \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&-1\\ 0&0&1&0 \end{pmatrix}$$

② $[K_2, K_3] = -J_1$

$$\begin{pmatrix} 0&0&-1&0\\ 0&0&0&0\\ -1&0&0&0\\ 0&0&0&0 \end{pmatrix}\begin{pmatrix} 0&0&0&-1\\ 0&0&0&0\\ 0&0&0&0\\ -1&0&0&0 \end{pmatrix} - \begin{pmatrix} 0&0&0&-1\\ 0&0&0&0\\ 0&0&0&0\\ -1&0&0&0 \end{pmatrix}\begin{pmatrix} 0&0&-1&0\\ 0&0&0&0\\ -1&0&0&0\\ 0&0&0&0 \end{pmatrix}$$

$$= \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&1\\ 0&0&0&0 \end{pmatrix} - \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&1&0 \end{pmatrix} = \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&1\\ 0&0&-1&0 \end{pmatrix}$$

$[J_2, K_3] = K_1$

$$\begin{pmatrix} 0&0&0&0\\ 0&0&0&+1\\ 0&0&0&0\\ 0&-1&0&0 \end{pmatrix}\begin{pmatrix} 0&0&0&-1\\ 0&0&0&0\\ 0&0&0&0\\ -1&0&0&0 \end{pmatrix} - \begin{pmatrix} 0&0&0&-1\\ 0&0&0&0\\ 0&0&0&0\\ -1&0&0&0 \end{pmatrix}\begin{pmatrix} 0&0&0&0\\ 0&0&0&1\\ 0&0&0&0\\ 0&-1&0&0 \end{pmatrix}$$

$$= \begin{pmatrix} 0&0&0&0\\ -1&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix} - \begin{pmatrix} 0&1&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}$$

$$= \begin{pmatrix} 0&-1&0&0\\ -1&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}$$

(b) Prove the commutation rule for $k=2,\ l=3$

$$\begin{aligned}[J_2, J_3] &= [-x^3\partial_1 + x^1\partial_3,\ -x^1\partial_2 + x^2\partial_1] \\ &= [x^3\partial_1,\ x^1\partial_2] + [x^1\partial_3 + x^2\partial_1] \\ &= x^3\partial_2 - x^2\partial_3 \\ &= J_1 \end{aligned}$$

$$\begin{aligned}[K_2, K_3] &= [x^0\partial_2 + x^2\partial_0,\ x^0\partial_3 + x^3\partial_0] \\ &= [x^0\partial_2,\ x^3\partial_0] + [x^2\partial_0,\ x^0\partial_3] \\ &= -x^3\partial_2 + x^2\partial_3 \\ &= -J_1 \end{aligned}$$

$$\begin{aligned}[J_2, K_3] &= [-x^3\partial_1 + x^1\partial_3,\ x^0\partial_3 + x^3\partial_0] \\ &= -[x^3\partial_1,\ x^0\partial_3] + [x^1\partial_3,\ x^3\partial_0] \\ &= x^0\partial_1 + x^1\partial_0 \\ &= K_1 \end{aligned}$$

3. (a) Show $\partial_\mu F^{\mu\nu} = 4\pi j^\mu$ is equivalent to inhomogenous maxwell equation.

The property of $F^{\mu\nu}$: $\begin{cases} F^{k0} = -F^{0k} = E^k \\ F^{mn} = -\epsilon_{mnl} B^l \end{cases}$ $(k.m.n.l = 1.2.3)$

① $\nu = 0$

$$\partial_\mu F^{\mu 0} = \partial_0 F^{00} + \partial_k \partial^{k0} = \partial_k E^k = \nabla \cdot \vec{E} = 4\pi j^0 = 4\pi \rho$$

② $\nu = m$, $(m = 1.2.3)$.

$$\partial_\mu F^{\mu m} = \partial_0 F^{0m} + \partial_l F^{lm}$$

$$= -\partial_0 E^m + (-\epsilon_{lmn}) \partial_l B^n$$

$$= -\partial_0 E^m + \epsilon_{lnm} \partial_l B^n$$

$$= 4\pi j^m$$

It is equivalent to say: $-\partial_t \vec{E} + \nabla \times \vec{B} = 4\pi \vec{j}$

From ① and ②, we can say that $\partial_\mu F^{\mu\nu} = 4\pi j^\nu$ is equivelent to $\begin{cases} \nabla \cdot \vec{E} = 4\pi \rho \\ \nabla \times \vec{B} = \partial_t \vec{E} + 4\pi \vec{j} \end{cases}$, which are the inhomogenous maxwell equations.

(b) Show $\partial^\sigma F^{\mu\nu} + \partial^\mu F^{\nu\sigma} + \partial^\nu F^{\sigma\mu} = 0$ is equivalent to the homogenous Maxwell Eq.

Clearly $\partial^\mu F^{\mu\mu} + \partial^\mu F^{\mu\mu} + \partial^\mu F^{\mu\mu} = 0$. If $\mu = \sigma$,

$$\partial^\sigma F^{\sigma\nu} + \partial^\sigma F^{\nu\sigma} + \partial^\nu F^{\sigma\sigma} = \partial^\sigma F^{\sigma\nu} - \partial^\sigma F^{\sigma\nu} = 0.$$

There are only 2 non-trivial equation due to the rotational symmetry of the Eq.

① $\sigma . \mu , \nu = 1. 2. 3.$

$$\begin{aligned} & \partial^1 F^{23} + \partial^2 F^{31} + \partial^3 F^{12} \\ &= \partial_1 B^1 + \partial_2 B^2 + \partial_3 B^3 \\ &= \nabla \cdot \vec{B} \\ &= 0 \end{aligned}$$

② $\sigma = 0$, $\mu . \nu = i, j$. with $i, j \in (1. 2. 3)$ and $i \neq j$.

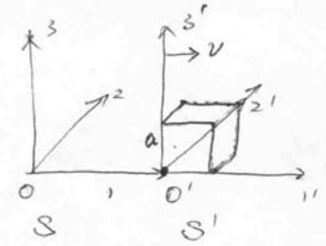
$$\begin{aligned} & \partial^0 F^{ij} + \partial^i F^{j0} + \partial^j F^{0i} \\ &= \partial_0 B^k (-\varepsilon_{ijk}) - \partial_i E^j + \partial_j E^i \\ &= 0 \end{aligned}$$

Multiply $-\epsilon_{ijk}$ on both sides:

$$\partial_0 B^k + \epsilon_{ijk} (\partial_i E^j - \partial_j E^i) = \partial_0 B^k + (\nabla \times \vec{E})_k = 0$$

Combine ① and ②: $\begin{cases} \nabla \cdot \vec{B} = 0 \\ \nabla \times \vec{E} = -\partial_t \vec{B} . \end{cases}$

4. (a)



In frame S:

$$\Delta x^{\mu} = \begin{pmatrix} 0 \\ \Delta x^1 \\ \Delta x^2 \\ \Delta x^3 \end{pmatrix}$$

in frame S'

$$\Delta x'^{\mu} = \begin{pmatrix} \Delta x^{0'} \\ a \\ a \\ a \end{pmatrix}$$

$$\begin{pmatrix} \Delta x^{0'} \\ a \\ a \\ a \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1-v^2}} & -\frac{v}{\sqrt{1-v^2}} & & \\ -\frac{1}{\sqrt{1-v^2}} & \frac{1}{\sqrt{1-v^2}} & & 0 \\ & & 1 & \\ & 0 & & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \Delta x^1 \\ \Delta x^2 \\ \Delta x^3 \end{pmatrix}$$

$$\begin{cases} \Delta x^1 = \sqrt{1-v^2}\, a \\ \Delta x^2 = a \\ \Delta x^3 = a \end{cases}$$

① $v = 0$, $\Delta x^1 = a$

② $v = \frac{1}{10}$, $\Delta x' = 0.9949\, a$

③ $v = \frac{1}{2}$, $\Delta x' = 0.8660\, a$

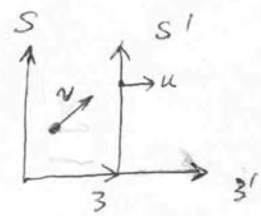
④ $v = \frac{9}{10}$, $\Delta x' = 0.4359\, a$

Considering the visual effect, the cube should also be twisted.

(b)

$$\begin{pmatrix} p^{0'} \\ p^{1'} \\ p^{2'} \\ p^{3'} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1-u^2}} & 0 & 0 & -\frac{u}{\sqrt{1-u^2}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{u}{\sqrt{1-u^2}} & 0 & 0 & \frac{1}{\sqrt{1-u^2}} \end{pmatrix} \begin{pmatrix} p^0 \\ p^1 \\ p^2 \\ p^3 \end{pmatrix}$$

$$\begin{cases} p'^0 = \frac{1}{\sqrt{1-u^2}} p^0 - \frac{u}{\sqrt{1-u^2}} p^3 \\ p'^1 = p^1 \\ p'^2 = p^2 \\ p'^3 = -\frac{u}{\sqrt{1-u^2}} p^0 + \frac{1}{\sqrt{1-u^2}} p^3 \end{cases}$$



To prove the addition theorem, devide p'^0 on both sides

$$\begin{cases} p'^1/p'^0 = v'^1 \\ p'^2/p'^0 = v'^2 \\ p'^3/p'^0 = v'^3 \end{cases} \qquad \begin{cases} \dfrac{p^1}{\frac{p^0}{\sqrt{1-u^2}} - \frac{up^3}{\sqrt{1-u^2}}} = \dfrac{\sqrt{1-u^2}\, v^1}{1-uv^3} \\ \dfrac{p^2}{\frac{p^0}{\sqrt{1-u^2}} - \frac{up^3}{\sqrt{1-u^2}}} = \dfrac{\sqrt{1-u^2}\, v^2}{1-uv^3} \\ \dfrac{-\frac{u}{\sqrt{1-u^2}} p^0 + \frac{1}{\sqrt{1-u^2}} p^3}{\frac{p^0}{\sqrt{1-u^2}} - \frac{up^3}{\sqrt{1-u^2}}} = \dfrac{-u+v^3}{1-uv^3} \end{cases}$$

$$\Rightarrow \begin{cases} v'^1 = \sqrt{1-u^2}\, \dfrac{v^1}{1-uv^3} \\ v'^2 = \sqrt{1-u^2}\, \dfrac{v^2}{1-uv^3} \\ v'^3 = \dfrac{v^3-u}{1-uv^3} \end{cases}$$

(c) Verify 10.107,

$$F'^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -\frac{E_y - v_1 B_z}{\sqrt{1-v_1^2}} & -\frac{E_z + v_1 B_y}{\sqrt{1-v_1^2}} \\ E_x & 0 & -\frac{B_z - v_1 E_y}{\sqrt{1-v_1^2}} & \frac{B_y + v_1 E_z}{\sqrt{1-v_1^2}} \\ \frac{E_y - v_1 B_z}{\sqrt{1-v_1^2}} & \frac{B_z - v_1 E_y}{\sqrt{1-v_1^2}} & 0 & -B_x \\ \frac{E_z + v_1 B_y}{\sqrt{1-v_1^2}} & -\frac{B_y + v_1 E_z}{\sqrt{1-v_1^2}} & B_x & 0 \end{pmatrix},$$

Tensor transforms as $F'^{\mu\nu} = L^{\mu}{}_{\alpha} L^{\mu}{}_{\beta} F^{\alpha\beta}$

$$F' = L F (L^T).$$

$$L = \begin{pmatrix} \gamma & -\beta & 0 & 0 \\ -\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{with} \quad \text{definitions} \begin{cases} \gamma \equiv \frac{1}{\sqrt{1-v_1^2}} \\ \beta \equiv \frac{v_1}{\sqrt{1-v_1^2}} \end{cases}$$

$$F = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}.$$

As $L = L^T$, $F' = LFL$.

$$FL = \begin{pmatrix} \beta E_x & -\gamma E_x & -E_y & -E_z \\ \gamma E_x & -\beta E_x & -B_z & B_y \\ \gamma E_y - \beta B_z & \gamma B_z - \beta E_y & 0 & -B_x \\ \gamma E_z + \beta B_y & -\gamma B_y - \beta E_z & B_x & 0 \end{pmatrix}$$

$$F' = LFL = \begin{pmatrix} \gamma\beta E_x - \cancel{\gamma\beta E_x} & -\gamma^2 E_x + \beta^2 E_x & -\gamma E_y + \beta B_z & -\gamma E_z - \beta B_y \\ \gamma^2 E_x - \beta^2 E_x & \cancel{-\gamma\beta E_x + \gamma\beta E_x} & -\gamma B_z + \beta E_y & \gamma B_y + \beta E_z \\ \gamma E_y - \beta B_z & \gamma B_z - \beta E_y & 0 & -B_x \\ \gamma E_z + \beta B_y & -\gamma B_y - \beta E_z & B_x & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -E_x & -\gamma(E_y - v_1 B_z) & -\gamma(E_z + v_1 B_y) \\ E_x & 0 & -\gamma(B_z - v_1 E_y) & \gamma(B_y + v_1 E_z) \\ \gamma(E_y - v_1 B_z) & \gamma(B_z - v_1 E_y) & 0 & -B_x \\ \gamma(E_z + v_1 B_y) & -\gamma(B_y + v_1 E_z) & B_x & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -E_x & -\frac{E_y - v_1 B_z}{\sqrt{1 - v_1^2}} & -\frac{E_z + v_1 B_y}{\sqrt{1 - v_1^2}} \\ E_x & 0 & -\frac{B_z - v_1 E_y}{\sqrt{1 - v_1^2}} & \frac{B_y + v_1 E_z}{\sqrt{1 - v_1^2}} \\ \frac{E_y - v_1 B_z}{\sqrt{1 - v_1^2}} & \frac{B_z - v_1 E_y}{\sqrt{1 - v_1^2}} & 0 & -B_x \\ \frac{E_z + v_1 B_y}{\sqrt{1 - v_1^2}} & -\frac{B_y + v_1 E_z}{\sqrt{1 - v_1^2}} & B_x & 0 \end{pmatrix}$$

(d) In the moving frame, $\vec{B} = 0$.

$$E_x = \frac{\lambda x}{2\pi\epsilon (x^2+y^2)}, \quad E_y = \frac{\lambda y}{2\pi\epsilon (x^2+y^2)}, \quad E_z = 0$$

The wire moves along the x^3 axis, so we have:

$E'_z = 0, \quad B'_z = 0.$

$$\begin{cases} E'_x = \dfrac{E_x}{\sqrt{1-v^2}}, \\ E'_y = \dfrac{E_y}{\sqrt{1-v^2}}, \end{cases} \qquad \begin{cases} B'_x = -\dfrac{v E_y}{\sqrt{1-v^2}} \\ B'_y = \dfrac{v E_x}{\sqrt{1-v^2}} \end{cases}$$

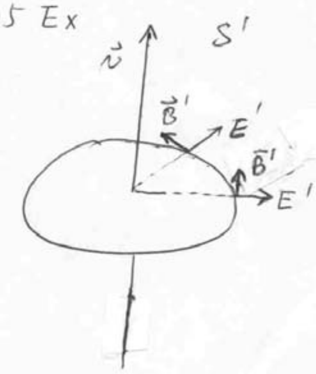
① $v = 0$, $\vec{E}' = \vec{E}$, $\vec{B}' = \vec{B} = 0$

② $v = \frac{1}{10}$, $\begin{cases} E'_x = 1.005 E_x \\ E'_y = 1.005 E_y, \end{cases} \quad \begin{cases} B'_x = -0.1005 E_y \\ B'_y = 0.1005 E_x \end{cases}$

③ $v = \frac{1}{2}$, $\begin{cases} E'_x = 1.155 E_x, \\ E'_y = 1.155 E_y, \end{cases} \quad \begin{cases} B'_x = -0.577 E_y \\ B'_y = 0.577 E_x \end{cases}$

④ $v = \frac{9}{10}$, $\begin{cases} E'_x = 2.294 E_x, \\ E'_y = 2.294 E_y, \end{cases} \quad \begin{cases} B'_x = -2.065 E_y \\ B'_y = 2.065 E_x \end{cases}$

Put them together, $\begin{cases} \vec{E}' = \dfrac{\vec{E}}{\sqrt{1-v^2}} \\ \vec{B}' = \vec{v} \times \vec{E}'. \end{cases}$



(e) In the rest frame, the wave function of pion is:

$$\psi_0(\vec{p}\,\lambda|x^\mu) = N e^{-ip\cdot x}, \quad \text{where } p\cdot x = p_\mu x^\mu.$$

The cross product is a scalar under Lorenz transformation. Therefore in the moving frame $\psi'(p'x') = \psi(p.x)$.

To find out the relation between p and p', we write

$$\psi'(p'x') = \psi(p.x)$$
$$= \psi(p, L^{-1}x').$$

Recall the definition of Ω: $\begin{cases} x'^0 = R\cosh\Omega \\ x'^3 = R\sinh\Omega \end{cases}$ and also

$$\begin{cases} \dfrac{1}{\sqrt{1-v^2}} = \cosh\omega^3 \\ \dfrac{v}{\sqrt{1-v^2}} = \sinh\omega^3. \end{cases}$$

$$\psi(p, L^{-1}x') = \hat{L}(\omega^3)\psi(p, x'(\Omega))$$
$$= \psi(p, x'(\Omega+\omega^3))$$
$$= N e^{-ip^0R\cosh(\Omega+\omega^3) + ip^3R\sinh(\Omega+\omega^3)}$$

Remember: $\cosh(\Omega+\omega^3) = \cosh\Omega\cosh\omega^3 + \sinh\Omega\sinh\omega^3$
$\sinh(\Omega+\omega^3) = \cosh\Omega\sinh\omega^3 + \sinh\Omega\cosh\omega^3.$

$$\psi(p, L^{-1}x') = N\exp\left[-i\mathcal{E}\left(\frac{x'^0}{\sqrt{1-v^2}} + x'^3\cdot\frac{v}{\sqrt{1-v^2}}\right) + ip^3\left(x'^0\frac{v}{\sqrt{1-v^2}} + x'^3\frac{1}{\sqrt{1-v^2}}\right)\right]$$
$$= N\exp\left[-i\left(\frac{\mathcal{E}}{\sqrt{1-v^2}} - \frac{p^3v}{\sqrt{1-v^2}}\right)x'^0 + i\left(\frac{p^3}{\sqrt{1-v^2}} - \frac{\mathcal{E}v}{\sqrt{1-v^2}}\right)x'^3\right]$$

We have then: $\begin{cases} \mathcal{E}' = \dfrac{1}{\sqrt{1-v^2}}(\mathcal{E} - p^3v) \\ p'^3 = \dfrac{1}{\sqrt{1-v^2}}(p - \mathcal{E}v) \end{cases}$ ——— (*)

Eq (*) is the same as the result in 4(b).