Problem Set 6
Physics 483 / Fall 2002
Professor Klaus Schulten

Problem 1: Hamiltonian and total charge of free Dirac field.

Lagrangian density of free Dirac field is $\mathcal{L}_D = \overline{\Psi}(\frac{i}{2}\gamma^\mu \overleftrightarrow{\partial_\mu} - mI)\Psi$, where $\overleftrightarrow{\partial_\mu}$ is defined as $\overline{\Psi} \overleftrightarrow{\partial_\mu} \Psi = \overline{\Psi}(\partial_\mu \Psi) - (\partial_\mu \overline{\Psi})\Psi$, I is the 4×4 identity matrix, and $\overline{\Psi} = \Psi^+ \gamma^0$. Using the quantilization formula,

$$\Psi(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3}} \frac{m}{k_0} \sum_{\alpha=1,2} [b_\alpha(\vec{k}) u^{(\alpha)}(\vec{k}) e^{-ikx} + d_\alpha^+(\vec{k}) v^{(\alpha)}(\vec{k}) e^{ikx}] \quad (1)$$

$$\Psi^+(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3}} \frac{m}{k_0} \sum_{\alpha=1,2} [b_\alpha^+(\vec{k}) \bar{u}^{(\alpha)}(\vec{k}) e^{ikx} + d_\alpha(\vec{k}) \bar{v}^{(\alpha)}(\vec{k}) e^{-ikx}], \quad (2)$$

and the anticommutation rules

$$\{b_\alpha(\vec{k}), b_\beta^+(\vec{k'})\}_{k_0=k'_0} = \frac{k_0}{m} \delta(\vec{k} - \vec{k'}) \delta_{\alpha\beta} \quad (3)$$

$$\{d_\alpha(\vec{k}), d_\beta^+(\vec{k'})\}_{k_0=k'_0} = \frac{k_0}{m} \delta(\vec{k} - \vec{k'}) \delta_{\alpha\beta} \quad (4)$$

$$everything \ else = 0, \quad (5)$$

show that the Hamiltonian and the total charge have the following forms:

$$\begin{aligned} \hat{H} &= \int d^3x : \Psi^+(x) i\partial_t \Psi(x) : \\ &= \int d^3k \frac{m}{k_0} k_0 \sum_\alpha [b_\alpha^+(\vec{k}) b_\alpha(\vec{k}) + d_\alpha^+(\vec{k}) d_\alpha(\vec{k})] \quad (6) \\ Q &= \int d^3x : j_0(x) : \\ &= \int d^3x : \Psi^+(x)\Psi(x) : \\ &= \int d^3k \frac{m}{k_0} \sum_\alpha [b_\alpha^+(\vec{k}) b_\alpha(\vec{k}) - d_\alpha^+(\vec{k}) d_\alpha(\vec{k})]. \quad (7) \end{aligned}$$

(Hint: To evaluate expressions such as $\frac{\partial}{\partial a}\bar{a}a$ and $\frac{\partial}{\partial \bar{a}}\bar{a}a$, we need specify the direction of ∂ operator, when the field operators a and $\bar{a}$ are not commutable such as for Fermions. Here we restrict ourselves to the following definitions: $\partial \equiv \overleftarrow{\partial}$, which means $\frac{\partial}{\partial a}\bar{a}a = \frac{\overleftarrow{\partial}}{\partial a}\bar{a}a = \bar{a}$, and $\frac{\partial}{\partial \bar{a}}\bar{a}a = \frac{\overleftarrow{\partial}}{\partial \bar{a}}\bar{a}a = -a$.)

Problem2: Derive equations of motion for Dirac particle coupled with electromagnetic field.

Given the Lagrangian densities

$$\mathcal{L} = \mathcal{L}_\gamma + \mathcal{L}_D + \mathcal{L}_I \quad (8)$$

$$\mathcal{L}_\gamma = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad (9)$$

$$\mathcal{L}_D = \sum_f \overline{\Psi^{(f)}(x)}(\frac{i}{2}\gamma^\mu \overleftrightarrow{\partial_\mu} - m_f I)\Psi^{(f)}(x) \quad (10)$$

$$\mathcal{L}_I = -e\sum_f \overline{\Psi^{(f)}(x)}\gamma^\mu\Psi^{(f)}(x)A_\mu(x), \quad (11)$$

show Euler Lagrangian equations lead to

$$(i\gamma^\mu\partial_\mu - m_f I)\Psi^{(f)}(x) = e\gamma^\nu\Psi^{(f)}(x)A_\nu(x) \quad (12)$$

$$\partial_\mu\partial^\mu A^\mu(x) = e\overline{\Psi^{(f)}(x)}\gamma^\mu\Psi^{(f)}(x) \quad (13)$$

Problem 3: Feynman propagator of EM field

Using the field operator expansion

$$A_\mu(x) = \frac{1}{\sqrt{(2\pi)^3}}\int\frac{d^3k}{2\omega_k}\sum_{\lambda=0}^{3}\epsilon_\mu^{(\lambda)}(k)[\hat{a}^{(\lambda)}(k)e^{-ikx} + \hat{a}^{(\lambda)+}(k)e^{ikx}], \quad (14)$$

and the commutation rules

$$[\hat{a}^{(\lambda)}(k), \hat{a}^{(\lambda')+}(k')] = -g^{\lambda\lambda'}2k_0\delta^3(\vec{k}-\vec{k}') \quad (15)$$

$$everything\ \ else = 0, \quad (16)$$

and also

$$\epsilon^{(\lambda)}\cdot\epsilon^{(\lambda)'} = g^{\lambda\lambda'} \quad (17)$$

to prove

$$< 0|T[A^\mu(x)A^\nu(y)]|0 >= \frac{i}{(2\pi)^4}\int d^4k\ e^{-ik(x-y)}\frac{-g^{\mu\nu}}{k^2+i\epsilon}, \quad (18)$$

where ϵ is an infinitesimal positive number.

Problem 4: Feynman propagator of Dirac particle

For Dirac particle, prove that

$$< 0|T[\Psi(x)\overline{\Psi(y)}]|0 >= \frac{i}{(2\pi)^4}\int d^4k\ e^{-ik(x-y)}\frac{k_\mu\gamma^\mu + mI}{k^2-m^2+i\epsilon}. \quad (19)$$

The problem set needs to be handed in by Thursday, December 12, 2002 into the mail box of Deyu Lu in Loomis.