Problem Set 4
Physics 483 / Fall 2002
Professor Klaus Schulten

Quantilization Klein-Gordon Fields:

The Lagrangian of the real KG field is $\mathcal{L}_{RKG} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$. We are going to study this Lagrangian in a detailed manner in the following problems.

Problem 1: Under the transformation $\vec{x} \to \vec{x} + \vec{a}$, where $\vec{a}$ is a constant vector, use Noether's theorem to prove the time independent conservative is $\vec{P} = -\int_{V\infty} d^3x \ \dot{\phi} \bigtriangledown \phi$.

Problem 2: Prove the integral $\int_\Omega d^4x \ \partial_\mu[\phi_n^* \overleftrightarrow{\partial^\mu} \phi_m]$ can be also written as $\int_\Omega d^4x \ \{[(\partial_\mu\partial^\mu + m^2)\phi_n^*]\phi_m - \phi_n^*[\partial_\mu\partial^\mu + m^2]\phi_m\}$.

Problem 3: Given $f_{\vec{k}} = \frac{1}{\sqrt{(2\pi)^3 2E(k)}} e^{-ik_\mu x^\mu}$, and $E(k) = k_0 = \sqrt{\vec{k}^2 + m^2}$, prove

(1) $< f_{\vec{k}}|f_{\vec{k}'} >= \delta(\vec{k} - \vec{k}')$,
(2) $< f_{\vec{k}}|f^*_{\vec{k}'} >= 0$,
(3) $< f^*_{\vec{k}}|f_{\vec{k}'} >= 0$.

Problem 4: Prove that any function $g(x)$, $x \in \mathcal{R}^4$ can be expanded as

$$g(x) = \int d^3k \ \ f_{\vec{k}}(x) \ \alpha(\vec{k}),$$

with the coefficients

$$\alpha(\vec{k}) = (f_{\vec{k}}(x)|g(x))_{x^0}.$$

Problem 5: Show that the commutation rule $i \ [\hat{p}_j, \hat{q}_j] = \delta_{ij}$ is consistent with $\hat{q}_j$ multiplicative, and $\hat{p}_j = -i\partial_j$.

Problem 6: Define the energy momentum four vector $P = (\hat{H}, \hat{\vec{P}})$, where

$$\begin{aligned}\hat{H} &= \int d^3x \left\{\frac{1}{2}[\hat{\pi}^2(x) + (\bigtriangledown\hat{\phi}(x))^2 + m^2\hat{\phi}^2(x)]\right\} \\ \vec{P} &= -\int d^3x \ \hat{\pi}(x) \ \bigtriangledown \hat{\phi}(x),\end{aligned}$$

show for any $F(x^\mu)$

(1) $\partial_\mu \hat{F} = i \ [P_\mu, F]$,
(2) $F(x'^\mu) = e^{i\hat{P}_\mu(x'^\mu - x^\mu)} \ F(x^\mu)e^{-i\hat{P}_\mu(x'^\mu - x^\mu)}$.

Problem 7: Prove the commutation rules:

$$[\hat{a}(\vec{k}), \hat{a}(\vec{k}')]_{k^0=k'^0} = [\hat{a}^+(\vec{k}), \hat{a}^+(\vec{k}')]_{k^0=k'^0} = 0.$$

Problem 8: Using the expression $\hat{\phi}(x) = \int d^3k \ [f_{\vec{k}}(x)\hat{a}(\vec{k}) + f^*_{\vec{k}}(x)\hat{a}^+(\vec{k})]$ to show that the Hamiltonian and momentum can be quantilized as

(1) $\hat{H} = \frac{1}{2}\int d^3k \ E(k) \ [\hat{a}^+(\vec{k})\hat{a}(\vec{k}) + \hat{a}(\vec{k})\hat{a}^+(\vec{k})]$,
(2) $\vec{P} = \frac{1}{2}\int d^3k \ \vec{k} \ [\hat{a}^+(\vec{k})\hat{a}(\vec{k}) + \hat{a}(\vec{k})\hat{a}^+(\vec{k})]$.

The problem set needs to be handed in by Thursday, November 07, 2002 into the mail box of Deyu Lu in Loomis.