Problem Set 3
Physics 483 / Fall 2002
Professor Klaus Schulten

Problem 1: MIT Bag Model of Quark Confinement

Hadrons are strongly interacting particles. There are two major types of hadrons: mesons, which are bosons with integral spin and baryons, which are fermions with half-integral spin. Mesons are believed to be composed of a "valence" quark (q) and antiquark ($\bar{q}$) pair, surrounded by a "sea" of gluons and other $q\bar{q}$ pairs; baryons are supposedly composed of three valence quarks surrounded by a similar sea. Furthermore, quarks are believed to exist only in the combinations of quarks and antiquarks which exist in baryons and mesons. If we attempt to remove a single quark from such a combination, the energy grows with the distance between the quark and its neighbors, until the energy becomes so large that it is energetically favorable to create a $q\bar{q}$ pair and break the "string" connecting the quark to its neighbors. Because of this property of the forces which bind quarks together, quarks are said to be *confined.*

The MIT model is a very simple model for hadronic structure. Suppose the hadron occupies a spherical volume of radius R. If a quark is inside this volume, we assume its mass is small, and it may be taken to be zero. If the quarks gets outside, interactions with the neighboring quarks which make up the rest of the hadron are assumed to generate an infinite mass for the quark. Since this implies infinite energy, the quark will not penetrate outside of the hadronic volume.

To describe this model quantitively, we solve the Dirac equation for the ground state and first excited state with $V = 0$. The mass is $m = 0$ inside the hadronic volume and $m \to \infty$ outside it.

(a) We assume the wave function of stationary state has the same form as Eq.(10.410), $\Psi(\vec{r}, t) = e^{-iEt} \begin{pmatrix} \phi(\vec{r}) \\ \chi(\vec{r}) \end{pmatrix}$. We will now adopt a notation for ϕ and χ different from Eq. (10.420-10.421),

$$\begin{pmatrix} \phi(\vec{r}) \\ \chi(\vec{r}) \end{pmatrix} = \begin{pmatrix} if_1(r)\mathcal{Y}_{jm}(j+\frac{1}{2}, \frac{1}{2}|\hat{r}) \\ -g_1(r)\mathcal{Y}_{jm}(j-\frac{1}{2}, \frac{1}{2}|\hat{r}) \end{pmatrix} \tag{1}$$

$$\begin{pmatrix} \phi(\vec{r}) \\ \chi(\vec{r}) \end{pmatrix} = \begin{pmatrix} if_2(r)\mathcal{Y}_{jm}(j-\frac{1}{2}, \frac{1}{2}|\hat{r}) \\ -g_2(r)\mathcal{Y}_{jm}(j+\frac{1}{2}, \frac{1}{2}|\hat{r}) \end{pmatrix}. \tag{2}$$

Prove that the parity operator $P = \gamma^0 \hat{P}$ commutes with H, which is the Hamiltonian of a Dirac particle in a spherically symmetric potential $V(r)$ with $\vec{A} = 0$. $\hat{P}$ has the following property

$$\hat{P}g(\vec{r}) = g(-\vec{r}), \tag{3}$$

with $g(\vec{r})$ being an arbitrary spatial function. Verify that the above two states belong to different eigenstates of P.

(b) Use equations (10.411 – 10.414) to derive differential equations for f_1, g_1 and f_2, g_2.

(c) Take $V = 0$ and eliminate g_1 to derive secondary differential equation for f_1 and do the same for f_2.

(d) Show that f_1 and f_2 satisfy the differential equation of spherical Bessel functions

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2} + k_0^2\right] f_l(k_0 r) = 0. \tag{4}$$

Give your expressions of k_0^2 and l. What are your expected values for l? Are they the same as what you obtained?

(e) If $k_0^2 \geq 0$, the solutions of Eq.4 are $j_l(k_0 r)$ and $n_l(k_0 r)$. While if $k_0^2 = -K_0^2 < 0$, the solutions are $h_l^{(1)}(iK_0 r)$ and $h_l^{(2)}(iK_0 r)$. As $n_l(k_0 r)$ and $h_l^{(2)}(iK_0 r)$ have singular behavior at $r \to 0$ and $r \to \infty$, respectively, we must only take $j_l(k_0 r)$ and $h_l^{(1)}(iK_0 r)$ as our solution. Useful expressions of the spherical Bessel functions are:

$$\begin{aligned} j_0(x) &= \frac{\sin x}{x}, & h_0^{(1)}(ix) &= -\frac{e^{-x}}{x} \\ j_1(x) &= \frac{\sin x}{x^2} - \frac{\cos x}{x}, & h_1^{(1)}(ix) &= i\frac{e^{-x}}{x}(1+\frac{1}{x}) \\ &\cdots\cdots & &\cdots\cdots. \end{aligned} \tag{5}$$

Recursion relations are

$$\frac{2l+1}{x} f_l(x) = f_{l-1}(x) + f_{l+1}(x) \tag{6}$$

$$f_l'(x) = \frac{1}{2l+1}[l f_{l-1}(x) - (l+1) f_{l+1}(x)], \tag{7}$$

where $f_l(x)$ can be any of $j_l(x)$, $n_l(x)$ and $h_l(ix)$, and f' refers to the derivative of f with respect to its argument x. (In case you feel unfamiliar with the spherical Bessel functions, there is a good introduction in chapter 7 of the class notes or at http://phyastweb.la.asu.edu/phy501-shumway/2001/notes/lec25.pdf). If we denote f_1 and f_2 as f_l, show that

$$g_1(x) = \frac{\kappa}{m+E} f_{l-1}(x) \tag{8}$$

$$g_2(x) = -\frac{\kappa}{m+E} f_{l+1}(x), \tag{9}$$

where $x = \kappa r$, so that $\kappa = k_0$ when $k_0^2 > 0$, and $\kappa = iK_0$ when $k_0^2 < 0$.

(f) Show that the general solutions have the following form:

$$\begin{pmatrix} \phi(\vec{r}) \\ \chi(\vec{r}) \end{pmatrix} = \begin{pmatrix} i f_l(x) \\ \frac{\kappa}{m+E}\vec{\sigma}\cdot\hat{r} f_{l-1}(x) \end{pmatrix} \mathcal{Y}_{jm}(l, \frac{1}{2}|\hat{r}) \tag{10}$$

$$\begin{pmatrix} \phi(\vec{r}) \\ \chi(\vec{r}) \end{pmatrix} = \begin{pmatrix} i f_l(x) \\ -\frac{\kappa}{m+E}\vec{\sigma}\cdot\hat{r} f_{l+1}(x) \end{pmatrix} \mathcal{Y}_{jm}(l, \frac{1}{2}|\hat{r}). \tag{11}$$

(g) Now we are interested in the positive energy solution only. Also note that the ground state is characterized by $j = \frac{1}{2}$, $l = 0$. Which solution should you choose? What is its parity? Under the limit that $m = 0$ when $r < R$, and $m \to \infty$ when $r > R$, write down the wave function of the ground state. You do not need to normalize it.

(h) The ground state energy can be obtained by matching the wave function inside and outside the sphere at $r = R$. What is the ground state energy?

(Hint: You need a prefactor me^{mR} for the wave function outside the hadronic volume to avoid it vanishing everywhere outside the sphere with radius R.)

(i) The first excited state is characterized by $l = 1$ and $j = \frac{3}{2}$. What's its parity then? What's the energy?

(j) Plot the radial solutions of ground state and first excited state using $R = 1$ and $m = 50$ outside the sphere.

Problem 2: Relativistic Hydrogen-type Atom

Evaluate and plot the radial wave functions for the $2p_{\frac{1}{2}}$ and $2p_{\frac{3}{2}}$ states of hydrogen-type atoms using the expressions derived in the class notes.

The problem set needs to be handed in by Thursday, October 17, 2002 into the mail box of Deyu Lu in Loomis.