Problem Set 2
Physics 483 / Fall 2002
Professor Klaus Schulten

Problem 1: Energies and Wave Function for Pionic Atom

Consider the stationary bound states of a pion π^- (spin-0 particle with mass $m = 139.577$ MeV / c^2, same charge q as electron, i.e., $q = -e$) in the Coulomb field of a nucleus with charge Ze. According to the derivation presented in class the corresponding wave function can be separated

$$\Psi(\vec{r}, t) \;=\; \frac{1}{r}\, R_\ell(r)\, Y_{\ell m}(\theta, \phi) \exp\left(-\frac{i}{\hbar} \epsilon t \right) \tag{1}$$

where $R_\ell(r)$ obeys the Klein-Gordon equation

$$\left[\frac{d^2}{dr^2} \;-\; \frac{\lambda(\lambda+1)}{r^2} \;+\; \frac{2\epsilon Z\alpha}{\hbar c r} \;-\; \frac{(mc^2)^2 \;-\; \epsilon^2}{\hbar^2 c^2} \right] R_\ell(r) \;=\; 0 \tag{2}$$

with

$$\lambda \;=\; -\frac{1}{2} \;+\; \sqrt{\left(\ell + \frac{1}{2} \right)^2 \;-\; (Z\alpha)^2}\ . \tag{3}$$

$\alpha = e^2/\hbar c$ is the fine structure constant. Note that we have introduced in the formulas above $\hbar$ and c explicitly, i.e., we do **not** assume *natural units.*

(a) Introduce the scaled radial coordinate

$$\rho \;=\; 2 \frac{\sqrt{(mc^2)^2 \;-\; \epsilon^2}}{\hbar c}\, r \tag{4}$$

and show that this leads to the differential equation

$$\left[\frac{d^2}{d\rho^2} \;-\; \frac{\lambda(\lambda+1)}{\rho^2} \;+\; \frac{\nu}{\rho} \;-\; \frac{1}{4} \right] R_\ell(\rho) \;=\; 0 \tag{5}$$

for

$$\nu \;=\; \frac{\epsilon Z\alpha}{\sqrt{(mc^2)^2 \;-\; \epsilon^2}}\ . \tag{6}$$

(b) In analogy to the case of stationary bound states for the non-relativistic hydrogen atom one can argue that the radial wave function $R_\ell(r)$ should obey for $\rho \to \infty$

$$R_\ell(\rho) \;\sim\; e^{-\rho/2} \tag{7}$$

and for $\rho \to 0$

$$R_\ell(\rho) \;\sim \rho^{\lambda+1}\ . \tag{8}$$

Provide these arguments for the present case.

Assume for $R_\ell(r)$ then the functional form

$$R_\ell(\rho) \;=\; N\, \rho^{\lambda+1}\, e^{-\rho/2} \sum_{j=0}^{n'} a_j \rho^j \tag{9}$$

where N is a normalization constant.

(c) Why should n' be chosen finite for a stationary bound state?

(d) Show that for the coefficients a_j in (9) holds

$$a_{j+1} \; = \; \frac{a_j}{(j+1)} \, \frac{a+j}{b+j} \; , \quad j \; = \; 1, 2, \ldots \tag{10}$$

where $a \, = \, \lambda \, - \, \nu \, + \, 1$ and $b \, = \, 2\lambda \, + \, 2$.

(e) Prove that the condition $a_{n'+1} \, = \, 0$ implies that ϵ is

$$\epsilon \; = \; mc^2 \, \frac{1}{\sqrt{1 \, + \, \frac{(Z\alpha)^2}{(n'+\frac{1}{2}+[(\ell+\frac{1}{2})^2 \, - \, (Z\alpha)^2]^{\frac{1}{2}})^2}}} \; . \tag{11}$$

(f) Show that for the energy $E_{n\ell}$, $n \, = \, n' + \ell + 1$ of the system associated with the ϵ-values in (11) the following expansion holds

$$E_{n\ell} \; = \; mc^2 \left[1 \, - \, \frac{(Z\alpha)^2}{2n^2} \, - \, \frac{(Z\alpha)^4}{2n^4} \left(\frac{n}{\ell + \frac{1}{2}} \, - \, \frac{3}{4} \right) \, + \, \ldots \right] \, . \tag{12}$$

(g) Plot $E_{n\ell}$ as a function of Z for $n \, = \, 1, \ell \, = 0$ and for $n \, = \, 2, \ell \, = 1$. What do you observe for $Z\alpha \, > \, \frac{1}{2}$ and for $Z\alpha \, > \, \frac{3}{2}$?

(h) Plot the wave functions $R_{n,\ell}$ for $n \, = \, 1, \ell \, = 0$ and for $n \, = \, 2, \ell \, = 1$ and $Z \, = \, 1, 50$. Compare the wave functions with the corresponding non-relativistic wave functions, employing for the latter wave functions expressions provided in standard textbooks. You are not required to determine the normalization constant in (9), but plot the wave functions on similar scales for better comparison.

Problem 2: Solution of the Dirac Equation for Particles Moving in Arbitrary Direction

Show that the wave function for a free spin$-\frac{1}{2}$ particle moving in arbitrary direction is

$$\Psi(\vec{p}, \lambda, \sigma|\vec{r}.t) \; = \; \sqrt{\frac{E_p \, + \, m}{2m}} \, e^{i(\vec{p}\cdot\vec{r} \, - \, \lambda E_p t)} \begin{cases} \omega_1(\vec{p}) & \text{for } \lambda = +, \, \sigma = +\frac{1}{2} \\ \omega_2(\vec{p}) & \text{for } \lambda = +, \, \sigma = -\frac{1}{2} \\ \omega_3(\vec{p}) & \text{for } \lambda = -, \, \sigma = +\frac{1}{2} \\ \omega_4(\vec{p}) & \text{for } \lambda = -, \, \sigma = -\frac{1}{2} \end{cases} \tag{13}$$

where $E_p \, = \, \sqrt{m^2 \, + \, p^2}$ and

$$\omega_1(\vec{p}) \; = \; \begin{pmatrix} 1 \\ 0 \\ \frac{p_3}{E+m} \\ \frac{p_+}{E+m} \end{pmatrix} \; ; \; \omega_2(\vec{p}) \; = \; \begin{pmatrix} 0 \\ 1 \\ \frac{p_-}{E+m} \\ \frac{-p_3}{E+m} \end{pmatrix} \; ;$$

$$\omega_3(\vec{p}) \; = \; \begin{pmatrix} \frac{-p_3}{E+m} \\ \frac{-p_+}{E+m} \\ 1 \\ 0 \end{pmatrix} \; ; \; \omega_4(\vec{p}) \; = \; \begin{pmatrix} \frac{-p_-}{E+m} \\ \frac{p_3}{E+m} \\ 0 \\ 1 \end{pmatrix} \; .$$

where we defined $p_\pm \, = \, p_1 \, \pm ip_2$.

Problem 3: Wave Packet of Spin$-\frac{1}{2}$ Particle at Rest

Consider a spin$-\frac{1}{2}$ particle described at time $t = 0$ through the wave function

$$\Psi(\vec{r}, t=0) \;=\; \left(\frac{1}{\pi d^2}\right)^{\frac{3}{4}} e^{-r^2/2d^2}\, \omega_1(\vec{p}=0) \tag{14}$$

where $\omega_1(\vec{p})$ has been defined in Problem 2.
(a) Determine the solution consistent with this initial condition through the expansion ($\omega_j(\vec{p})$ as defined in Problem 1)

$$\int d^3p \sum_{j=1}^{4} b_j(\vec{p})\, \omega_j(\vec{p})\, e^{i(\vec{p}\cdot\vec{r} \,-\, \lambda_j E_p t)} \;; \quad \lambda_{1,2} \;=\; +1,\; \lambda_{3,4} \;=\; -1\;. \tag{15}$$

Here $b_j(\vec{p})$ are scalar functions (expansion coefficients).
(b) Evaluate the quantities

$$\begin{aligned} R_+ &= \frac{|b_1(\vec{p})|^2 \;+\; |b_2(\vec{p})|^2}{N(\vec{p})} \\ R_- &= \frac{|b_3(\vec{p})|^2 \;+\; |b_4(\vec{p})|^2}{N(\vec{p})} \\ N(\vec{p}) &= |b_1(\vec{p})|^2 \;+\; |b_2(\vec{p})|^2 \;+\; |b_3(\vec{p})|^2 \;+\; |b_4(\vec{p})|^2, \end{aligned}$$

discuss how $R_\pm$ depends on d and plot $R_\pm$ as a function of p for the following cases: (1). $m = 1$, $d = 10$; (2). $m = 1$, $d = 1$; (3). $m = 1$, $d = 0.1$.
(c) Determine the densities

$$|b_j(\vec{p})|^2\, \omega_j(\vec{p})^\dagger\, \gamma^0\, \omega_j(\vec{p}) \;; \qquad j \;=\; 1,\, 2,\, 3,\, 4 \tag{16}$$

and interprete the result.
(d) Compare the results in (b) and (c) with the corresponding behaviour of a non-relativistic spin$-\frac{1}{2}$ particle with initial state

$$\psi(\vec{r}, t=0) \;=\; \left(\frac{1}{\pi d^2}\right)^{\frac{3}{4}} e^{-r^2/2d^2}\, |\frac{1}{2}, +\frac{1}{2}\rangle\;. \tag{17}$$

The problem set needs to be handed in by Thursday, October 3, 2002 into the mail box of Deyu Lu in Loomis.