Take Home Final Exam, Version III
Physics 483 / Fall 2002
Professor Klaus Schulten

Please contact deyulu@ks.uiuc.edu if there are any questions or errors. Please come back regularly to the website and check for revised versions of the exam.

Promised notes on the Feynman rules will be on the website on Sunday.

You are expected to complete the take home exam completely on your own. You can use textbooks and class notes, but you cannot consult any person in reaching your solutions.

Problem 1: Relativistic Landau levels

Assume a homogeneous field $\vec{B}$ along the z axis, the potential vector $\vec{A}$ of which is chosen such that $A^0 = A^x = A^z = 0$, and $A^y = Bx$. Determine stationary states $\psi = e^{-iEt}\begin{pmatrix} \phi \\ \chi \end{pmatrix}$ for Dirac particles moving in this field.

(a) By eliminating χ, show

$$(E^2 - m^2)\phi = [(\vec{p})^2 + e^2B^2x^2 - eB(\sigma_z + 2xp_y)]\phi. \tag{1}$$

(b) As p_y, p_z and σ_z commute with the right hand side, seek the solution of the form

$$\phi(\vec{x}) = e^{i(p_y y + p_z z)}\, f(x). \tag{2}$$

Derive the differential equation for $f(x)$.

(c) Assume $eB > 0$ and show that the differential equation reduces to

$$(-\frac{d^2}{d\xi^2} + \xi^2 - \sigma_z)f = af. \tag{3}$$

State what ξ and a are in your equation.

(d) Take f_α as eigen-vectors of σ_z such that

$$\sigma_z f_\alpha = \alpha f_\alpha, \qquad \alpha = \pm 1. \tag{4}$$

The solution f_α is a Hermite polynomial $H_n(\xi)$, with $f_\alpha = ce^{-\xi^2/2}H_n(\xi)$. Provided $a + \alpha = 2n + 1$, calculate the energy spectrum.

Problem 2: Atom in a perfect cavity.

Consider a cavity without loss that has a fundamental eigen-frequency ω_o and real eigen-mode $f(x)$ with polarization vector $\vec{e}$. Disregarding the other cavity modes the field in the cavity is described by the operator for the vector potential

$$\hat{\vec{A}}(\vec{x}, t) \;=\; \sqrt{\frac{\hbar}{2\omega_o}}\,\vec{e}\,\left[\, f(\vec{x})e^{i\omega_0 t}\,\hat{a}^\dagger \;+\; f(\vec{x})e^{-i\omega_0 t}\,\hat{a}\,\right] . \tag{5}$$

An atom is placed in this cavity and undergoes transitions between its ground state $|\alpha\rangle$ and excited state $|\beta\rangle$ having energies E_α and E_β. The other transitions are neglected. To first order in the coupling constant and using the dipolar approximation the Hamiltonian of the atom plus photon system is

$$H \;=\; H_0 \;+\; H_I \tag{6}$$

where

$$H_0 \;=\; \hbar\omega_o\, \hat{a}^\dagger \hat{a} \;+\; E_\alpha |\alpha\rangle\langle\alpha| \;+\; E_\beta |\beta\rangle\langle\beta| \tag{7}$$

and

$$H_I \;=\; (\,\hat{a}^\dagger \;+\; \hat{a}\,)\,(\,g_o\,|\alpha\rangle\langle\beta| \;+\; g_o\,|\beta\rangle\langle\alpha|\,) \tag{8}$$

with g_o representing the coupling constant. One splits

$$H \;=\; H_I^{\mathrm{res}} \;+\; H_I^{\mathrm{nres}} \tag{9}$$

where

$$H_I^{\mathrm{res}} \;=\; g_o\,\hat{a}^\dagger\,|\alpha\rangle\langle\beta| \;+\; g_o\,\hat{a}\,|\beta\rangle\langle\alpha| \tag{10}$$

is the resonant coupling that describes the process of excitation (deexcitation) of the atom by absorption / emission of a photon.

(a) Diagonalize exactly the Hamiltonian $H^{\mathrm{res}} \;=\; H_0 \;+\; H_I^{\mathrm{res}}$ and find the eigenvalues and normalized eigenvectors for states that include n photons.

(b) Show that the transition probability between the (unperturbed) states $|$atom state α, one photon$\rangle$ and $|$atom state β, no photon$\rangle$ is

$$p_{1\to 2}(t) \;=\; \sin^2\left(\frac{|g_o|\,t}{\hbar}\right) \tag{11}$$

(c) Assume that the cavity is prepared in a state $|n\rangle$ that has n photons. Consider now that we place a transparent mirror into the middle of the cavity. Let $a_{\mathrm{L}}^\dagger$ and $a_{\mathrm{R}}^\dagger$ create the photon on the left and right side of the mirror with $\hat{a}^\dagger \;=\; \frac{1}{\sqrt{2}}\,(\hat{a}_{\mathrm{L}}^\dagger \;+\; \hat{a}_{\mathrm{R}}^\dagger)$. What is the probability q_s to find the state $|n\rangle$ in a state $|s\rangle_L$ that has s photons on the left side of the mirror. Verify $\sum_{s=0}^{n} q_s = 1$. (Do not spend too much time on this part of the problem.)

Problem 3: A simple field theory

Consider a real field $\hat{\phi}(x,t)$ confined to a one-dimensional box of length L, i.e. $x \in [0, L]$, $\hat{\phi}(x,t) = 0$ for $x = 0, L$. The field is characterized through the Lagrangian density $\mathcal{L}(\hat{\phi}, \partial_x\hat{\phi}, \partial_t\hat{\phi})$

$$\mathcal{L} = \frac{1}{2}(\partial_t\hat{\phi})^2 - \frac{1}{2}(\partial_x\hat{\phi})^2, \tag{12}$$

and the commutation property

$$[\hat{\phi}(x,t), \hat{\pi}(y,t)] \;=\; i\,\delta(x-y) \tag{13}$$

$$[\hat{\phi}(x,t), \hat{\phi}(y,t)] \;=\; 0 \tag{14}$$

$$[\hat{\pi}(x,t), \hat{\pi}(y,t)] \;=\; 0, \tag{15}$$

where $\hat{\pi}$ is the momentum conjugate of $\hat{\phi}$, i.e.,

$$\hat{\pi} = \frac{\partial\mathcal{L}}{\partial(\partial_t\hat{\phi})}. \tag{16}$$

Develop the quantilization of $\hat{\phi}$.

(a) Consider first the classical field ϕ. Establish the solutions of the Euler Lagrangian equation of the form

$$\phi(x,t) = \phi_n(x)e^{\pm i\omega_n t}, \tag{17}$$

where

$$\int_0^L dx\, \phi_n(x)\, \phi_m(x) = \delta_{nm}. \tag{18}$$

(b) Expand the field $\hat{\phi}(x,t)$ in terms of operators a_n and $a_n^\dagger$,

$$\hat{\phi}(x,t) = \sum_n \frac{1}{\sqrt{2\omega_n}}[\hat{a}_n\phi_n(x)e^{-i\omega_n t} + \hat{a}_n^\dagger\phi_n(x)e^{i\omega_n t}]. \tag{19}$$

Using Eq.13 prove that $\hat{a}_n$ and $\hat{a}_n^\dagger$ should obey the commutation properties

$$\begin{aligned} [\hat{a}_n, \hat{a}_m^\dagger] &= \delta_{nm} && (20) \\ [\hat{a}_n, \hat{a}_m] &= 0 && (21) \\ [\hat{a}_n^\dagger, \hat{a}_m^\dagger] &= 0. && (22) \end{aligned}$$

(c) Defining the Hamiltonian through

$$\hat{H} = \int_0^L dx\mathcal{H}(x,t), \tag{23}$$

and

$$\mathcal{H} = \hat{\pi}\dot{\hat{\phi}} - \mathcal{L}, \tag{24}$$

express H in terms of $\hat{a}_n$ and $\hat{a}_n^\dagger$.

(d) Using $\hat{a}_n^\dagger$ construct the stationary states of $\hat{H}$.

Problem 4: Emission of hydrogen atoms

Hydrogen atoms with density N are initially prepared in the $n = 3$, $l = 2$, $m = 0$ state. For the sake of simplicity, we ignore spin. Describe the radiation in the non-relativistic limit within the dipole approximation.

(a) What are the selection rules?

(b) What is/are the emitted photon energy/energies?

(c) What is the overall rate of z-polarized emission?

(d) What is the angular distribution of z-polarized emission?

Problem 5: Electron-positron scattering

In an electron storage ring electrons and positrons are scattered in collinear beams with opposite momenta and create a muon/ antimuon pair: $e^- + e^+ \to \mu^- + \mu^+$. Assume that the kinetic energy is much larger then both electron and muon masses such that in the following one can take the ratio masses / $E \to 0$. Describe the scattering in leading approximation through the single diagram shown in Figure 1.

In determining the transition consider initial spin states with spin directions along the directions of the initial momenta defined through $\mathrm{f}_{L,R}^{\mathrm{charge}}$ where, e.g., e_R^+ denotes a positron with spin in the direction of the momentum and μ_L^- denotes a muon with spin in the direction opposite to the momentum. The wave functions corresponding to the examples are denoted by v_R and u_L. In (a) and (b) we consider the Dirac spinor part of the states, not yet applying the Feynman rules, e.g., not using $\overline{v_R}$ for an incoming antiparticle.

(a) Show that in the high energy approximation the initial states for $\mathrm{e}_L^- + \mathrm{e}_R^+$ scattering are

$$u_L^{(i)}(p_-) = \sqrt{E(p_-)} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \tag{25}$$

$$v_R^{(i)}(p_+) = \sqrt{E(p_+)} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} . \tag{26}$$

(b) The final particles (muon / anti-muon) have again opposite momenta. Define the angle between the incoming and outgoing momenta as θ. Show that the final states in the channel $\mu_L^- + \mu_R^+$ can be expressed

$$u_L^{(f)}(q_-) = \sqrt{E(q_-)} \begin{pmatrix} -\sin(\theta/2) \\ \cos(\theta/2) \\ \sin(\theta/2) \\ -\cos(\theta/2) \end{pmatrix} \tag{27}$$

$$v_R^{(f)}(q_+) = \sqrt{E(q_+)} \begin{pmatrix} -\cos(\theta/2) \\ -\sin(\theta/2) \\ \cos(\theta/2) \\ \sin(\theta/2) \end{pmatrix} . \tag{28}$$

(c) State the expression for T_{fi} for the proper initial and final states for $\mathrm{e}_L^- + \mathrm{e}_R^+ \to \mu_L^- + \mu_R^+$ scattering, i.e., split off a factor $i\,(2\pi)^4\,\delta(p_- + p_+ - q_- - q_+)$ from R_{fi} as given by the Feynman rules.

(d) Show $T_{fi} = e^2(1 + \cos\theta)$.

(e) Show that for $\mathrm{e}_L^- + \mathrm{e}_R^+ \to \mu_R^- + \mu_L^+$ scattering one obtains $T_{fi} = e^2(1 - \cos\theta)$.

(f) Show that for $\mathrm{e}_L^- + \mathrm{e}_L^+$ in the initial channel one obtains $T_{fi} = 0$.

The take home exam needs to be placed by Saturday, December 21,

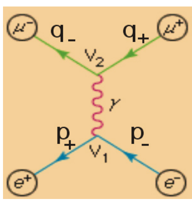

Figure 1: electron-positron scattering.

2002 at noon into the Loomis mail box of Deyu Lu in Loomis.