Quantum Wave Packet Moving in Harmonic Potential

Consider a one-dimensional quantum-mechanical harmonic oscillator described by the Schrödinger equation

$$i\hbar\partial_t\psi(x,t) \;=\; \left(-\frac{\hbar^2}{2m}\partial_x^2 \;+\; \frac{1}{2}m\omega^2 x^2\right)\psi(x,t)\;. \tag{1}$$

The system is assumed to be in the initial state

$$\psi(x,t_o) \;=\; \left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}} \exp\left[-\frac{m\omega}{2\hbar}(x \,-\, x_o)^2\right] \tag{2}$$

(a) Introduce the variables

$$X \;=\; \sqrt{\frac{m\omega}{\hbar}}\,x\;, \qquad T \;=\; \omega t \tag{3}$$

and show that (1, 2) become, defining $X_o \;=\; \sqrt{m\omega/\hbar}\,x_o,\; T_o \;=\; \omega\, t_o,$

$$i\partial_T\Psi(X,T) \;=\; \frac{1}{2}\left(-\partial_X^2 \;+\; X^2\right)\Psi(X,T) \tag{4}$$

and

$$\Psi(X,T_o) \;=\; \pi^{-\frac{1}{4}}\exp\left[-\frac{1}{2}(X - X_o)^2\right] \tag{5}$$

(b) Show that the solution to (4, 5) is

$$\Psi(X,T) \;=\; \pi^{-\frac{1}{4}}\exp\left\{-\frac{1}{2}[X - X_o(T)]^2 \;+\; iXK_o(T) \;-\; i\Phi_o(T) \;-\; \frac{i}{2}(T - T_o)\right\} \tag{6}$$

where

$$\begin{aligned} X_o(T) &= X_o\cos(T - T_o) \\ K_o(T) &= -\,X_o\sin(T - T_o) \\ \Phi_o(T) &= -\,\frac{1}{4}X_o^2\sin 2(T - T_o)\;. \end{aligned} \tag{7}$$

(c) Following the notebook supplied for problem 1 above plot $|\Psi(X,T)|^2$ as a function of X at times $T \,=\, T_o \,+\, \frac{1}{4}n\pi,\; n \,=\, 0, 1, \ldots 8.$ Assume $X_o \,=\, 1.$

(d) Discuss the motion of the oscillator as observed in (c).