Project SU(3)
Physics 481 / Spring 2000
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Problem 1 : Lie Algebra of the Group SU(3)

Consider the commutation properties of the eight generators of the group SU(3)

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad , \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad , \quad \lambda_4; = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$
$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad , \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$
$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad , \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Using *Mathematica* verify the commutation properties of these matrices

$$[\lambda_j, \, \lambda_k] = 2\, i\, f_{jk\ell}\, \lambda_\ell \tag{1}$$
$$[\lambda_j, \, \lambda_k]_+ = \frac{4}{3}\delta_{jk}\mathbb{1} \; + \; 2\, d_{jk\ell}\, \lambda_\ell \; . \tag{2}$$

Employ the matrix notation of *Mathematica*, e.g., for the matrix λ_1 above

$$l1 = \{\{0,1,0\},\{1,0,0\},\{0,0,0\}\} \tag{3}$$

and use the built-in matrix product ".", e.g., "$l1.l2$". Also note that matrix elements are addressed in *Mathematica* like l1[[1,2]]. Furthermore, to evaluate the trace of a matrix use a command like Sum[l1[[i,i]], {i,1,3}].

(a) Verify that the following property holds

$$\mathrm{tr}\,(\lambda_j \lambda_k) = 2\, \delta_{jk} \; . \tag{4}$$

(b) Using (1, 2) demonstrate that $f_{jk\ell}$ is a totally anti-symmetric tensor, i.e., any transposition of two indices yields the same value, except for a change of sign.

(c) Using (1, 4) demonstrate that $d_{jk\ell}$ is a totally symmetric tensor, i.e., for any transposition of two indices holds $d_{jk\ell} = d_{kj\ell}$ etc.

(d) Verify the table of non-vanishing $f_{jk\ell}$ and $d_{jk\ell}$ given in class. For this purpose employ the compact expression of these constants introduced in (b,c).

(e) Demonstrate that the following two matrices (so-called Casimir operators) commute with all generators λ_j:

$$\begin{aligned} C_1 &= \sum_{j=1}^{8} \lambda_j^2 \\ C_2 &= \sum_{j,k,\ell} d_{jk\ell}\, \lambda_j\, \lambda_k\, \lambda_\ell \ . \end{aligned} \tag{5}$$

Problem 2: Matrix Representations of Quark and Anti-Quark Multiplets

(a) Consider the SU(3) multiplet [3] describing the family of "u", "d", and "s" quarks. Express the action of the operators $T_{\pm,3}$, $U_{\pm,3}$, $V_{\pm,3}$ on the states of the quark multiplet through appropriate 3×3–matrices.

(b) Using the relationships which connect the operators $T_{\pm,3}$, $U_{\pm,3}$, $V_{\pm,3}$ with the operators F_j, $j = 1, 2, \ldots, 8$

$$\begin{aligned} T_\pm &= F_1 \pm iF_2 \ , \qquad T_3 = F_3 \\ V_\pm &= F_4 \pm iF_5 \ , \qquad U_\pm = F_6 \pm iF_7 \\ V_3 &= \tfrac{3}{4}Y + \tfrac{1}{2}T_3 \ , \qquad U_3 = \frac{3}{4}Y - \frac{1}{2}T_3 \ , \quad Y = \frac{2}{\sqrt{3}}F_8 \end{aligned} \tag{6}$$

and $F_j = \frac{1}{2}\lambda_j$ $j = 1, 2, \ldots, 8$ express λ_j in terms of 3×3–matrices.

(c) Repeat (a) and (b) for the multiplet $\overline{[3]}$ representing the anti-quarks.

Problem 3: Quark Wave Functions of Spin-$\frac{3}{2}$ Baryons

Baryons are formed from three quarks. States composed of three quarks are elements of the representation $[3] \otimes [3] \otimes [3]$ where [3] is the representation D(1,0) of SU(3) corresponding to the quarks u, d, s. The quarks also carry spin–$\frac{1}{2}$. The states of $[3] \otimes [3] \otimes [3]$, including the spin–$\frac{1}{2}$ attributes, will be denoted by

$$\begin{aligned} u_\downarrow d_\uparrow s_\uparrow &= u(1)|\frac{1}{2}, -\frac{1}{2}\rangle_1 \, d(2)|\frac{1}{2}, \frac{1}{2}\rangle_2 \, s(3)|\frac{1}{2}, -\frac{1}{2}\rangle_3 \quad , \\ u_\uparrow u_\uparrow s_\downarrow &= u(1)|\frac{1}{2}, \frac{1}{2}\rangle_1 \, u(2)|\frac{1}{2}, \frac{1}{2}\rangle_2 \, s(3)|\frac{1}{2}, -\frac{1}{2}\rangle_3 \end{aligned} \tag{7}$$

etc. The representation $[3] \otimes [3] \otimes [3]$ of SU(3) can be expressed in terms of the irreducible representation D(3,0)=[10], two D(1,1)=[8] and a D(0,0)=[1], i.e.,

$$[3] \otimes [3] \otimes [3] \; = \; [10] \oplus [8] \oplus [8] \oplus [1] \; . \tag{8}$$

In the following we will consider first the representation [10], called the baryon decuplet, and then one of the representations [8], called the baryon octet. The first representation contains the Δ particles, the particle which appears as a resonance in nucleon–pion scattering, the second representation contains the two nucleons, i.e., the proton and the neutron.

The baryon octet [10] contains an isospin quartet ($I = \frac{3}{2}$) of Δ's, namely, $\Delta^{++}, \Delta^{+}, \Delta^{0}, \Delta^{-}$, an isospin triplet ($I = 1$) of Σ^*'s, namely, $\Sigma^{*+}, \Sigma^{*0}, \Sigma^{*-}$, an isospin doublet ($I = \frac{1}{2}$) of Ξ^*'s, namely, Ξ^{*0}, Ξ^{*-}, and finally an isospin singlet ($I = 0$), namely, the Ω^-. All these 'particles' have spin $\frac{3}{2}$.

(a) The wave function of the Δ^{++} is

$$|\Delta^{++}\rangle \; = \; u_\uparrow u_\uparrow u_\uparrow \; . \tag{9}$$

Using the total spin operators

$$S_{\pm,3} \; = \; {}^{(1)}S_{\pm,3} + {}^{(2)}S_{\pm,3} + {}^{(3)}S_{\pm,3} \tag{10}$$

show that the spin state of $|\Delta^{++}\rangle$ is $|\frac{3}{2}, \frac{3}{2}\rangle$.

(b) What are the eigenvalues of the total isospin and total hypercharge operators

$$T_3 \; = \; {}^{(1)}T_3 + {}^{(2)}T_3 + {}^{(3)}T_3 \quad , \; Y \; = \; {}^{(1)}Y + {}^{(2)}Y + {}^{(3)}Y \tag{11}$$

of $|\Delta^{++}\rangle$? Here the operators ${}^{(j)}O$ denote the operators acting on the j-th quark. Show that the quantum numbers are in agreement with the total charge of +2 of the Δ^{++} by applying the charge operator $\hat{Q} = \frac{1}{2}Y + T_3$.

(c) Applying the operators

$$\begin{aligned} T_\pm &= {}^{(1)}T_\pm + {}^{(2)}T_\pm + {}^{(3)}T_\pm \\ V_\pm &= {}^{(1)}V_\pm + {}^{(2)}V_\pm + {}^{(3)}V_\pm \\ U_\pm &= {}^{(1)}U_\pm + {}^{(2)}U_\pm + {}^{(3)}U_\pm \end{aligned} \tag{12}$$

generate, starting from $|\Delta^{++}\rangle$ as given in (9), all other states of the baryon decouplet. Use for this purpose ($j = 1, 2, 3$)

$$\begin{aligned} {}^{(j)}T_- u(j) &= d(j) \quad , \quad {}^{(j)}T_+ d(j) = u(j) \\ {}^{(j)}V_- u(j) &= s(j) \quad , \quad {}^{(j)}V_+ s(j) = u(j) \\ {}^{(j)}U_- d(j) &= s(j) \quad , \quad {}^{(j)}U_+ s(j) = d(j) \end{aligned} \tag{13}$$

and the fact that all other actions of these operators lead to 'zero', e.g., ${}^{(j)}V_- d(j) = 0$. Normalize the resulting states and assign the quantum numbers t_3, y of the operators T_3 and Y as defined in (11).

(d) Provide the wave function which describes the Δ^+ in the spin state $|\frac{3}{2}, \frac{1}{2}\rangle$.

(e) Using (13) determine the representation of the operators $V_\pm$, $U_\pm$ and $T_\pm$ corresponding to [10]. Note that these operators, in case of representation [10], are 10×10 matrices. You should exercise care in accounting properly for the normalization factors of the states which will modify the respective matrix elements. Provide also the representations of the operators T_3 and Y using the quantum numbers t_3, y of all states determined in (c).

(f) Determine the matrices λ_j, $j = 1, \ldots 8$ corresponding to the operators in (d) employing the relationships $\lambda_1 = T_1 + T_-$, etc. (Use for this tedious job and the remaining part of (f) *Mathematica*.) Note that the λ_j's are also 10×10–matrices. Test, if the matrices obey the Lie algebra of the group SU(3), i.e., $[\lambda_j, \lambda_k] = 2i\, f_{jk\ell} \lambda_\ell$.

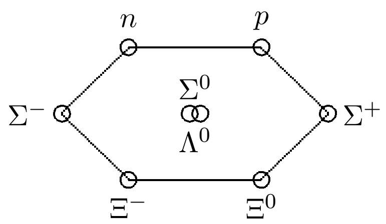
Problem 4: Quark Wave Functions of Spin-$\frac{1}{2}$ Baryons

(a) Consider now the baryon octet [8]. The neutron is an element of this representation with t_3, y quantum numbers $-\frac{1}{2}, 1$. The wave function of the neutron in the spin state $|\frac{1}{2}, \frac{1}{2}\rangle$ is

$$\begin{aligned} |n\rangle \;=\; \frac{1}{\sqrt{18}} \;\; & (\, 2d_\uparrow u_\downarrow d_\uparrow \, + \, 2d_\uparrow d_\uparrow u_\downarrow \, + \, 2u_\downarrow d_\uparrow d_\uparrow \\ & - d_\uparrow u_\uparrow d_\downarrow \, - \, d_\uparrow d_\downarrow u_\uparrow \, - \, u_\uparrow d_\downarrow d_\uparrow \\ & - \, d_\downarrow u_\uparrow d_\uparrow \, - \, d_\downarrow d_\uparrow u_\uparrow \, - \, u_\uparrow d_\uparrow d_\downarrow \,) \; . \end{aligned} \tag{14}$$

Verify that the neutron, thus described, has spin $\frac{1}{2}$. Construct from the neutron state the state of the proton by applying the operator T_+ as defined in (12).

(b) Construct similarly the wave functions of all states on the perimiter of the [8] multiplet:



Octet of Spin-$\frac{1}{2}$ Baryons

(c) Verify that the six states constructed in (b) form two T-doublets, two V-doublets, and two U-doublets with respect to the respective total $T, V, U-$ operators as defined in (12).

(d) Construct now the two central states Σ^0, Λ^0 proceeding as follows: First construct the state $\Sigma^0 \;=\; |1,\, 0\rangle_T$ by applying T_- defined in (12) to the perimeter state $\Sigma^+ \;=\; |1,\, 1\rangle_T$ (see figure). Then construct the state $|1,\, 0\rangle_V$ describing the central state of a V triplet by applying V_- to the perimeter state $p \;=\; |1,\, 1\rangle_V$. Show that the resulting state is not orthogonal to the state $\Sigma^0 \;=\; |1,\, 0\rangle_T$ constructed already. Obtain a proper orthogonal state.

(e) Show that the state constructed through application of U_+ to the perimeter state Ξ^0 can be expressed in terms of the two states constructed in (d). Show this explicitly by expressing the state $U_+\Xi^-$ as a linear combination of the states constructed in (d). Demonstrate this property also by considering the commutator $[U_+, V_-]$.

This project needs to be handed in by Friday, May 12, 2000 into the mail box of Gheorghe-Sorin Paraoan in Loomis.