Problem Set 6
Physics 481 / Spring 2000
Professor Klaus Schulten

Problem 1: Solution of the Dirac Equation for Particles Moving in Arbitrary Direction

Show that the wave function for a free spin–$\frac{1}{2}$ particle moving in arbitrary direction is

$$\Psi(\vec{p}, \lambda, \sigma|\vec{r}.t) \;=\; \sqrt{\frac{E_p \,+\, m}{2m}}\; e^{i(\vec{p}\cdot\vec{r}\,-\,\lambda E_p t)} \begin{cases} \omega_1(\vec{p}) & \text{for } \lambda = +,\ \sigma = +\frac{1}{2} \\ \omega_2(\vec{p}) & \text{for } \lambda = +,\ \sigma = -\frac{1}{2} \\ \omega_3(\vec{p}) & \text{for } \lambda = -,\ \sigma = +\frac{1}{2} \\ \omega_4(\vec{p}) & \text{for } \lambda = -,\ \sigma = -\frac{1}{2} \end{cases} \tag{1}$$

where $E_p \;=\; \sqrt{m^2 \,+\, p^2}$ and

$$\omega_1(\vec{p}) \;=\; \begin{pmatrix} 1 \\ 0 \\ \frac{p_3}{E+m} \\ \frac{p_+}{E+m} \end{pmatrix} ;\ \omega_2(\vec{p}) \;=\; \begin{pmatrix} 0 \\ 1 \\ \frac{p_-}{E+m} \\ \frac{-p_3}{E+m} \end{pmatrix} ;$$

$$\omega_3(\vec{p}) \;=\; \begin{pmatrix} \frac{p_3}{E+m} \\ \frac{p_+}{E+m} \\ 1 \\ 0 \end{pmatrix} ;\ \omega_4(\vec{p}) \;=\; \begin{pmatrix} \frac{p_-}{E+m} \\ \frac{-p_3}{E+m} \\ 0 \\ 1 \end{pmatrix} .$$

where we defined $p_\pm \;=\; p_1 \,\pm\, i p_2$.

Problem 2: Wave Packet of Spin–$\frac{1}{2}$ Particle at Rest

Consider a spin–$\frac{1}{2}$ particle described at time $t \,=\, 0$ through the wave function

$$\Psi(\vec{r}, t = 0) \;=\; \left(\frac{1}{\pi d^2}\right)^{\frac{3}{4}} e^{-r^2/2d^2} \omega_1(\vec{p} = 0) \tag{2}$$

where $\omega_1(\vec{p})$ has been defined in Problem 1.
(a) Determine the solution consistent with this initial condition through the

expansion ($\omega_j(\vec{p})$ as defined in Problem 1)

$$\int d^3p \sum_{j=1}^{4} b_j(\vec{p})\,\omega_j(\vec{p})\,e^{i(\vec{p}\cdot\vec{r}\,-\,\lambda_j E_p t)}\ ; \quad \lambda_{1,2} \,=\, +1,\ \lambda_{3,4} \,=\, -1\ . \tag{3}$$

Here $b_j(\vec{p})$ are scalar functions (expansion coefficients).
(b) Evaluate the quantities

$$\begin{aligned} R_+ &= \frac{|b_1(\vec{p})|^2 \,+\, |b_2(\vec{p})|^2}{N(\vec{p})} \\ R_- &= \frac{|b_3(\vec{p})|^2 \,+\, |b_4(\vec{p})|^2}{N(\vec{p})} \\ N(\vec{p}) &= |b_1(\vec{p})|^2 \,+\, |b_2(\vec{p})|^2 \,+\, |b_3(\vec{p})|^2 \,+\, |b_4(\vec{p})|^2 \end{aligned}$$

and express the result as a function of $n \,=\, \sqrt{1 \,+\, (d_C/d)^2}$ where $d_C \,=\, m^{-1}$ is the Compton length. Plot $R_\pm$ as a function of n and discuss it.
(c) Determine the densities

$$|b_j(\vec{p})|^2\,\omega_j(\vec{p})^\dagger\,\gamma^0\,\omega_j(\vec{p})\ ; \qquad j \,=\, 1,\,2,\,3,\,4 \tag{4}$$

and interprete the result.
(d) Compare the results in (b) and (c) with the corresponding behaviour of a non-relativistic spin$-\frac{1}{2}$ particle with initial state

$$\psi(\vec{r}, t=0) \,=\, \left(\frac{1}{\pi d^2}\right)^{\frac{3}{4}} e^{-r^2/2d^2}\ |\frac{1}{2},\, +\frac{1}{2}\rangle\ . \tag{5}$$

Problem 3: Dirac Particle in One-Dimensional Square Well Potential

Determine the stationary solution of the Dirac equation

$$(\,i\gamma^\mu(\partial_\mu \,+\, ieA_\mu) \,-\, m\,)\ \Psi(x^\nu) \,=\, 0 \tag{6}$$

for $A_\mu \,=\, (V(x^1), 0, 0, 0)$ where

$$V(x^1) \,=\, \begin{cases} 0 & |x^1| \,>\, \frac{a}{2} \\ -V_o & |x^1| \le \frac{a}{2} \end{cases}\ , \qquad V_o \,>\, 0 \tag{7}$$

Consider only bound states.
(a) Solve first the non-relativistic problem to familiarize yourself with the well-known solution in this case.
(b) State the form of the stationary wave functions in the relativistic case and express the Dirac equation in the form $i\partial_t \Psi = H_D \Psi$.
(c) State the general solutions in the three regions $x^1 < -\frac{a}{2}$, $-\frac{a}{2} \leq x^1 \leq \frac{a}{2}$, and $\frac{a}{2} < x^1$.
(d) Employ the fact that the wave functions are continuous at $x^1 = \pm\frac{a}{2}$ to obtain the solution, in particular, the spectrum. As in the non-relativistic case the spectrum cannot be given in closed analytical form, however, a graphical procedure for obtaining the spectrum can be formulated. Argue why in case of the Dirac equation it does not hold necessarily that the derivatives of wave functions are continuous at $x^1 = \pm\frac{a}{2}$.
(e) Discuss the result, in particular, the behaviour of the solution for $V_o < 2m$ and for $V_o > 2m$.

Due Thursday, April 13 in mail box of Gheorghe-Sorin Paraoan.