Problem Set 5
Physics 481 / Spring 2000
Professor Klaus Schulten

Problem 1: Non-Interacting Electron Gas

Consider a system of $2N$ non-interacting electrons inside a cube with sides L.

(a) Prove that the one-electron normalized eigenstates, using periodic boundary conditions, are given by

$$\psi_{\vec{k}} = \frac{1}{\sqrt{V}} e^{i\vec{k}\vec{r}} \quad \text{and} \quad \epsilon_{\vec{k}} = \frac{\hbar^2 k^2}{2m} \tag{1}$$

where $V = L^3$ is the volume of the cube and the possible values of the wave vector $\vec{k}$ are

$$k_x = \frac{2\pi}{L} n_x \ , \quad k_y = \frac{2\pi}{L} n_y \ , \quad k_z = \frac{2\pi}{L} n_z \tag{2}$$

where n_x,n_y and n_z are integers ranging from $-\infty$ to $+\infty$.

(b) Relate the energy eigenstates given by (1) to those obtained by using boundary conditions which assume that $\psi(\vec{r})$ vanishes on the inner surface of the box.

(c) Show that the number of states (1) with $\vec{k} \in (\vec{k}_0, \vec{k}_0 + \Delta\vec{k})$ is given by

$$\Delta N = 2 \frac{V \Delta k_x \Delta k_y \Delta k_z}{(2\pi)^3} \ , \tag{3}$$

where the extra factor of 2 accounts for the two possible spin orientations.

Note that if N is very large ($\sim 10^{23}$) then we can write (3) in the following differential form

$$dN = 2 \frac{V d^3 k}{(2\pi)^3} \ , \qquad d^3 k = dk_x dk_y dk_z \ . \tag{4}$$

(d) The one-particle density of states (DOS) is defined by

$$\varrho(\epsilon) = \frac{dN}{d\epsilon_{\vec{k}}}(\epsilon) \ , \tag{5}$$

i.e., DOS represents the ratio between the number of states dN within the energy interval $(\epsilon, \epsilon + d\epsilon)$ and the

length $d\epsilon$ of this interval.

Prove that for a non-interacting electron gas

$$\varrho(\epsilon) = \sqrt{2}\frac{m^{\frac{3}{2}}V}{\pi^2\hbar^3}\sqrt{\epsilon} \ . \tag{6}$$

(e) According to the Pauli principle, the ground state, i.e., the state with the lowest total energy, of the non-interacting electron gas is obtained by filling up all the states (1) with $k < k_F$ (the Fermi momentum) or, equivalently, with $\epsilon < \epsilon_F$ (the Fermi energy)[1].

Calculate k_F using particle conservation, i.e.,

$$2N = \int_{k<k_F} dN \stackrel{(4)}{=} 2\int_{k<k_F} \frac{Vd^3k}{(2\pi)^3} \ . \tag{7}$$

Similarly, determine ϵ_F by solving the equation

$$2N = \int_{\epsilon<\epsilon_F} dN \stackrel{(5-6)}{=} \sqrt{2}\int_{\epsilon<\epsilon_F} \frac{m^{\frac{3}{2}}V}{\pi^2\hbar^3}\sqrt{\epsilon}d\epsilon \ . \tag{8}$$

You can check the correctness of your results via the formula $\epsilon_F = \frac{\hbar^2 k_F^2}{2m}$.

Problem 2: Lorentz Transformations

(a) Consider a cylinder oriented along the x_3-axis of diameter 2 and length 4 moving along (i) the x_1-axis and (ii) along the x_2-axis with the velocities (in units of c) $v = 0, \frac{1}{10}, \frac{1}{2}, \frac{9}{10}$. Plot the cylinder as it appears to an observer in a frame at rest. Use the `mathematica` command

$$ParametricPlot3D[Sin[t], Cos[t], u, t, 0, 2Pi, u, 0, 4]$$

[1] The filled states in the $\vec{k}$-space form a sphere, called the *Fermi-sea* or the *Fermi-sphere*.

(b) Using the transformation behaviour of the momentum 4-vector p^μ prove the addition theorem of velocities.

(c) A thin wire along the x_3-axis carries a charge density $\frac{q}{\ell}\,\delta(x_1)\,\delta(x_2)$. Determine the electrical and magnetic field in the rest frame when the wire moves along the x_3-axis with velocities (in units of c) $v = 0, \frac{1}{10}, \frac{1}{2}, \frac{9}{10}$.

Problem 3: Pion in δ–potential

Consider a pion, i.e., particle with spin zero and mass m_π, bound by a scalar potential of the form

$$V(x^\mu) \;=\; -V_o\,\delta(r)/4\pi r^2 \tag{9}$$

where r is the radial distance of the pion from the origin. Solve the Klein Gordon equation for the special case when the solution is static, i.e., independent of time. Discuss the significance of your result.

Problem 4: Pion in square well potential

A pion, i.e., a particle of spin zero and mass m_π is bound by a scalar one-dimensional square well potential $V(x^1)$ defined to be

$$V(x^1) \;=\; \begin{cases} 0 & R \,<\, x^1 \\ -m_\pi^2 V_o & 0 \,<\, x^1 \,<\, R \\ \infty & x \,<\, 0 \end{cases} \tag{10}$$

This could be a very rough model for a pion inside a nucleus of radius R.

(a) Solve the Klein-Gordon equation in one space dimension for the positive energy ground state.

(b) Find the value of R such that the positive energy ground state has energy

$$E_o \;=\; m_\pi \sqrt{1 \,-\, V_o/2} \tag{11}$$

Due Thursday, April 13 in mail box of Gheorghe-Sorin Paraoanu.