Solutions to Linear Algebra Quiz
Physics 480 / Fall 1999
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Problem 1: Inner Product

In the "conventional" notation the vectors $|1\rangle$, $|2\rangle$, $|1'\rangle$, $|2'\rangle$ can be defined as

$$|1\rangle \;=\; \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}\,, \quad |2\rangle \;=\; \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\,, \quad |1'\rangle \;=\; \begin{pmatrix} a_1' \\ a_2' \end{pmatrix}\,, \quad |2'\rangle \;=\; \begin{pmatrix} b_1' \\ b_2' \end{pmatrix}\,. \tag{1}$$

where

$$\begin{pmatrix} a_1' \\ a_2' \end{pmatrix} \;=\; \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \tag{2}$$

$$\begin{pmatrix} b_1' \\ b_2' \end{pmatrix} \;=\; \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\,. \tag{3}$$

The scalar products are

$$\langle 1|2\rangle \;=\; a_1^* b_1 \;+\; a_2^* b_2\,, \tag{4}$$

and similar for $\langle 1'|2'\rangle$. It follows, using

$$a_1' \;=\; \cos\phi\; a_1 \;-\; \sin\phi\; a_2 \tag{5}$$
$$a_2' \;=\; \sin\phi\; a_1 \;+\; \cos\phi\; a_2 \tag{6}$$
$$b_1' \;=\; \cos\phi\; b_1 \;-\; \sin\phi\; b_2 \tag{7}$$
$$b_2' \;=\; \sin\phi\; b_1 \;+\; \cos\phi\; b_2 \tag{8}$$

and the definition $c \;=\; \cos\phi$, $s \;=\; \sin\phi$,

$$\begin{aligned} \langle 1'|2'\rangle \;&=\; (ca_1 \;-\; sa_2)^*\,(cb_1 \;-\; sb_2) \;+\; (sa_1 \;+\; ca_2)^*\,(sb_1 \;+\; cb_2) \\ &=\; c^2\,a_1^* b_1 \;-\; sc\,a_2^* b_1 \;-\; cs\,a_1^* b_2 \;+\; s^2\,a_2 b_2 \\ &+\; s^2\,a_1^* b_1 \;+\; cs\,a_2^* b_1 \;+\; sc\,a_1^* b_2 \;+\; c^2\,a_2 b_2 \\ &=\; \left(c^2 \;+\; s^2\right) a_1^* b_1 \;-\; (sc \;-\; cs)\, a_2^* b_1 \\ &-\; (cs \;-\; sc)\, a_1^* b_2 \;+\; \left(s^2 \;+\; c^2\right) a_2^* b_2 \end{aligned} \tag{9}$$

From $c^2 \;+\; s^2 \;=\; 1$ and $sc \;-\; cs \;=\; 0$ follows the desired result.

Problem 2: Commutator

Using the definition $[L_1, L_2] = L_1 L_2 - L_2 L_1$ we obtain

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{10}$$

Problem 3: Function of Operator

The expression $\exp A$ is defined through the Taylor expansion of the exponential

$$\exp A = 1\!\!1 + A + \frac{1}{2} A^2 + \frac{1}{6} A^3 + \cdots \tag{11}$$

Using

$$A^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{12}$$

and the resulting property $A^n = 0$ for any $n > 1$, we obtain

$$\exp A = 1\!\!1 + A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} . \tag{13}$$

Problem 4: Matrix Diagonalization

The form of the matrix implies that

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{14}$$

is an eigenvector with eigenvalue $\lambda_1 = 2$. The remaining two eigenvectors are of the form

$$|2\rangle = \begin{pmatrix} 0 \\ a_1 \\ b_1 \end{pmatrix} , \quad |3\rangle = \begin{pmatrix} 0 \\ a_2 \\ b_2 \end{pmatrix} \tag{15}$$

where the components a_1, b_1, a_2, b_2 and the eigenvalues associated with $|2\rangle$, $|3\rangle$ are defined through the eigenvalue problem posed by the 2×2-matrix

$$\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} . \tag{16}$$

The normalized eigenvectors and associated eigenvalues of the latter problem are

$$|\tilde{2}\rangle \;=\; \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \;,\quad |\tilde{3}\rangle \;=\; \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{17}$$

$$\lambda_2 \;=\; 2\;,\quad \lambda_3 \;=\; 4\;. \tag{18}$$

We conclude that the 2nd eigenvector and 3rd eigenvector of the original problem are

$$|2\rangle \;=\; \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \;,\quad |3\rangle \;=\; \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \tag{19}$$

with the associated eigenvalues given by (18). We note that, due to the degeneracy $\lambda_1 = \lambda_2 = 2$, any linear combination $c_1|1\rangle + c_2|2\rangle$ is also an eigenvector with eigenvalue 2, but that there exist only two linearly independent eigenvectors with eigenvalue 2.

Problem 5: Function Space

A scalar product can be defined through

$$\langle n|m\rangle \;=\; \int_0^L dx\; \psi_n(x)\,\psi_m(x)\;. \tag{20}$$

It follows indeed, using $\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$,

$$\langle n|m\rangle \;=\; \frac{1}{L}\int_0^L dx\left[\cos\frac{(n-m)\pi x}{L} \;-\; \cos\frac{(n+m)\pi x}{L}\right] \tag{21}$$

or choosing as integration variable $y = x/L$

$$\langle n|m\rangle \;=\; \int_0^1 dy\,[\,\cos(n-m)\pi x \;-\; \cos(n+m)\pi x\,]\;. \tag{22}$$

In case of $n = m$ follows

$$\langle n|n\rangle \;=\; \int_0^1 dy\,[\,1 \;-\; \cos 2n\pi x\,] \;=\; 1 \;-\; \frac{1}{2n\pi}\,[\,\sin 2n\pi \;-\; \sin(0)\,] \;=\; 1\;. \tag{23}$$

In case of $n \neq m$ follows

$$\begin{aligned}\langle n|m\rangle \;&=\; \frac{1}{(n-m)\pi}\,[\,\sin(n-m)\pi \;-\; \sin(0)\,] \\ &\quad -\frac{1}{(n+m)\pi}\,[\,\sin(n+m)\pi \;-\; \sin(0)\,] \;=\; 0\;.\end{aligned} \tag{24}$$

Accordingly, the set of functions with the scalar product (21) is orthonormal and we can state

$$\langle n|m\rangle \;=\; \delta_{nm} \tag{25}$$

From $f(x) = \sum_m c_m \psi_m(x)$ follows

$$\begin{aligned}\int_0^L dx\ \psi_n(x)\, f(x) &= \sum_m c_m \int_0^L dx\ \psi_n(x)\, \psi_m(x) \\ &= \sum_m c_m\ \delta_{nm} = c_n\ . \end{aligned} \tag{26}$$

It follows

$$c_m = \left(\frac{2}{L}\right)^{\frac{1}{2}} \int_0^{L/2} dx\ \sin\frac{m\pi x}{L} = \frac{\sqrt{2L}}{m\pi}\ \sin\frac{m\pi}{2}\ . \tag{27}$$