Chapter 4

Linear Harmonic Oscillator

The linear harmonic oscillator is described by the Schrödinger equation

$$i\,\hbar\,\partial_t\,\psi(x,t) \;=\; \hat{H}\,\psi(x,t) \tag{4.1}$$

for the Hamiltonian

$$\hat{H} \;=\; -\,\frac{\hbar^2}{2m}\,\frac{\partial^2}{\partial x^2} \;+\; \frac{1}{2}m\,\omega^2 x^2\;. \tag{4.2}$$

It comprises one of the most important examples of elementary Quantum Mechanics. There are several reasons for its pivotal role. The linear harmonic oscillator describes vibrations in molecules and their counterparts in solids, the phonons. Many more physical systems can, at least approximately, be described in terms of linear harmonic oscillator models. However, the most eminent role of this oscillator is its linkage to the boson, one of the conceptual building blocks of microscopic physics. For example, bosons describe the modes of the electromagnetic field, providing the basis for its quantization. The linear harmonic oscillator, even though it may represent rather non-elementary objects like a solid and a molecule, provides a window into the most elementary structure of the physical world. The most likely reason for this connection with fundamental properties of matter is that the harmonic oscillator Hamiltonian (4.2) is symmetric in momentum and position, both operators appearing as quadratic terms.

We have encountered the harmonic oscillator already in Sect. 2 where we determined, in the context of a path integral approach, its propagator, the motion of coherent states, and its stationary states. In the present section we approach the harmonic oscillator in the framework of the Schrödinger equation. The important role of the harmonic oscillator certainly justifies an approach from two perspectives, i.e., from the path integral (propagotor) perspective and from the Schrödinger equation perspective. The path integral approach gave us a direct route to study time-dependent properties, the Schrödinger equation approach is suited particularly for stationary state properties. Both approaches, however, yield the same stationary states and the same propagator, as we will demonstrate below.

The Schrödinger equation approach will allow us to emphasize the algebraic aspects of quantum theory. This Section will be the first in which an algebraic formulation will assume center stage. In this respect the material presented provides an important introduction to later Sections using Lie algebra methods to describe more elementary physical systems. Due to the pedagodical nature of this Section we will link carefully the algebraic treatment with the differential equation methods used so far in studying the Schrödinger equation description of quantum systems.

In the following we consider first the stationary states of the linear harmonic oscillator and later consider the propagator which describes the time evolution of any initial state. The stationary states of the harmonic oscillator have been considered already in Chapter 2 where the corresponding wave functions (2.235) had been determined. In the framework of the Schrödinger equation the stationary states are solutions of (4.1) of the form $\psi(x,t) = \exp(-iEt/\hbar)\phi_E(x,t)$ where

$$\hat{H}\,\phi_E(x) \;=\; E\,\phi_E(x)\;. \tag{4.3}$$

Due to the nature of the harmonic potential which does not allow a particle with finite energy to move to arbitrary large distances, all stationary states of the harmonic oscillator must be bound states and, therefore, the natural boundary conditions apply

$$\lim_{x\to\pm\infty} \phi_E(x) \;=\; 0\;. \tag{4.4}$$

Equation (4.3) can be solved for any $E \in \mathbb{R}$, however, only for a discrete set of E values can the boundary conditions (4.4) be satisfied. In the following algebraic solution of (4.3) we restrict the Hamiltonian $\hat{H}$ and the operators appearing in the Hamiltonian from the outset to the space of functions

$$\mathcal{N}_1 \;=\; \{f:\mathbb{R}\to\mathbb{R},\, \mho \in \mathbb{C}_\infty,\, \lim_{\curvearrowleft\to\pm\infty} \mho(\curvearrowleft) \;=\; \nvdash\} \tag{4.5}$$

where C_∞ denotes the set of functions which together with all of their derivatives are continuous. It is important to keep in mind this restriction of the space, in which the operators used below, act. We will point out explicitly where assumptions are made which built on this restriction. If this restriction would not apply and all functions $f:\mathbb{R}\to\mathbb{R}$ would be admitted, the spectrum of $\hat{H}$ in (4.3) would be continuous and the eigenfunctions $\phi_E(x)$ would not be normalizable.

4.1 Creation and Annihilation Operators

The Hamiltonian operator (4.2) can be expressed in terms of the two operators

$$\hat{p} \;=\; \frac{\hbar}{i}\frac{d}{dx}\;, \quad \hat{x} \;=\; x \tag{4.6}$$

the first being a differential operator and the second a multiplicative operator. The operators act on the space of functions $\mathcal{N}_1$ defined in (4.5). The Hamiltonian $\hat{H}$ can be expressed in terms of the operators acting on the space (4.5)

$$\hat{H} \;=\; \frac{1}{2m}\hat{p}^2 \;+\; \frac{1}{2}m\,\omega^2\,\hat{x}^2 \tag{4.7}$$

which is why these operators are of interest to us.
The cardinal property exploited below, beside the representation (4.7) of the Hamiltonian, is the commutation property

$$[\hat{p},\,\hat{x}] \;=\; \frac{\hbar}{i}\,\mathbb{1} \tag{4.8}$$

which holds for any position and momentum operator. This property states that $\hat{p}$ and $\hat{x}$ obey an algebra in which the two do not commute, however, the commutator has a simple form. In order to derive (4.8) we recall that all operators act on functions in $\mathcal{N}_1$ and, hence, the action of $[\hat{p}, \hat{x}]$ on such functions must be considered. One obtains

$$[\hat{p}, \hat{x}]\, f(x) \;=\; \frac{\hbar}{i}\frac{d}{dx}\, x\, f(x) \;-\; x\,\frac{\hbar}{i}\frac{d}{dx}\, f(x) \;=\; \frac{\hbar}{i}\, f(x) \;=\; \frac{\hbar}{i}\, \mathbb{1}\, f(x)\;. \tag{4.9}$$

From this follows (4.8).

Our next step is an attempt to factorize the Hamiltonian (4.7) assuming that the factors are easier to handle than the Hamiltonian in yielding spectrum and eigenstates. Being guided by the identity for scalar numbers

$$(b - ic)(b + ic) \;=\; b^2 \;-\; i(cb - bc) \;+\; c^2 \;=\; b^2 \;+\; c^2 \tag{4.10}$$

we define

$$\hat{a}^+ \;=\; \sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} \;-\; \frac{i}{\sqrt{2m\hbar\omega}}\,\hat{p}\,, \quad \hat{a}^- \;=\; \sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} \;+\; \frac{i}{\sqrt{2m\hbar\omega}}\,\hat{p}\,. \tag{4.11}$$

The reader may note that we have attempted, in fact, to factor $\hat{H}/\hbar\omega$. Since $\hat{a}^+$ and $\hat{a}^-$ are operators and not scalars we cannot simply expect that the identy (4.10) holds for $\hat{a}^+$ and $\hat{a}^-$ since $[\hat{a}^-, \hat{a}^+] = \hat{a}^-\hat{a}^+ - \hat{a}^+\hat{a}^-$ does not necessarily vanish. In fact, the commutator property (4.8) implies

$$[\hat{a}^-, \hat{a}^+] \;=\; \mathbb{1}\;. \tag{4.12}$$

To prove this commutation property we determine using (4.11)

$$\hat{a}^-\hat{a}^+ \;=\; \frac{m\omega}{2\hbar}\,\hat{x}^2 \;+\; \frac{1}{2m\hbar\omega}\,\hat{p}^2 \;+\; \frac{i}{2\hbar}\,[\hat{p}, \hat{x}]\;. \tag{4.13}$$

(4.7) and (4.8) yield

$$\hat{a}^-\hat{a}^+ \;=\; \frac{1}{\hbar\omega}\,\hat{H} \;+\; \frac{1}{2}\,\mathbb{1}\;. \tag{4.14}$$

Similarly one can show

$$\hat{a}^+\hat{a}^- \;=\; \frac{1}{\hbar\omega}\,\hat{H} \;-\; \frac{1}{2}\,\mathbb{1}\;. \tag{4.15}$$

(4.14) and (4.15) together lead to the commutation property (4.12).

Before we continue we like to write (4.14, 4.15) in a form which will be useful below

$$\hat{H} \;=\; \hbar\omega\,\hat{a}^-\hat{a}^+ \;-\; \frac{\hbar\omega}{2}\,\mathbb{1} \tag{4.16}$$

$$\hat{H} \;=\; \hbar\omega\,\hat{a}^+\hat{a}^- \;+\; \frac{\hbar\omega}{2}\,\mathbb{1}\;. \tag{4.17}$$

We also express $\hat{a}^+$ and $\hat{a}^-$ directly in terms of the coordinate x and the differential operator $\frac{d}{dx}$

$$\hat{a}^+ \;=\; \sqrt{\frac{m\omega}{2\hbar}}\,x \;-\; \sqrt{\frac{\hbar}{2m\omega}}\,\frac{d}{dx}\,, \quad \hat{a}^- \;=\; \sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} \;+\; \sqrt{\frac{\hbar}{2m\omega}}\,\frac{d}{dx}\;. \tag{4.18}$$

It is of interest to note that the operators $\hat{a}^+$ and $\hat{a}^-$ are real differential operators.

Relationship between a^+ and a^-

The operators $\hat{a}^+$ and $\hat{a}^-$ are related to each other by the following property which holds for all functions $f,\, g \in \mathcal{N}_1$

$$\int_{-\infty}^{+\infty} dx\, f(x)\, a^+\, g(x) \;=\; \int_{-\infty}^{+\infty} dx\, g(x)\, a^-\, f(x)\;. \tag{4.19}$$

This property states that the operators $\hat{a}^+$ and $\hat{a}^-$ are the adjoints of each other. The property follows directly from (**??**). Using (**??**) we like to state (4.19) in the form[1]

$$\langle f|\hat{a}^+ g\rangle_{\Omega_\infty} \;=\; \overline{\langle g|\hat{a}^- f\rangle_{\Omega_\infty}}\;. \tag{4.20}$$

In the following we will determine the spectrum of $\hat{H}$ and its eigenstates. The derivation will be based solely on the properties (4.12, 4.16, 4.17, 4.19).

$\hat{a}^+$ and $\hat{a}^-$ as Ladder Operators

The operators $\hat{a}^+$ and $\hat{a}^-$ allow one to generate all stationary states of a harmonic oscillator once one such state $\phi_E(x)$

$$\hat{H}\,\phi_E(x) \;=\; E\,\phi_E(x) \tag{4.21}$$

is available. In fact, one obtains using (4.16, 4.17, 4.21)

$$\begin{aligned}
\hat{H}\,\hat{a}^-\,\phi_E(x) &= (\,\hbar\omega\,\hat{a}^-\hat{a}^+ \;-\; \frac{\hbar\omega}{2}\,\mathbb{1}\,)\,\hat{a}^-\,\phi_E(x)\\
&= \hat{a}^-\,(\,\hbar\omega\,\hat{a}^+\hat{a}^- \;+\; \frac{\hbar\omega}{2}\,\mathbb{1} \;-\; \hbar\omega\,\mathbb{1}\,)\,\phi_E(x)\\
&= \hat{a}^-\,(\,\hat{H} \;-\; \hbar\omega\,)\,\phi_E(x)\\
&= (\,E \;-\; \hbar\omega\,)\,\hat{a}^-\,\phi_E(x)\;.
\end{aligned} \tag{4.22}$$

Similarly one can show using again (4.16, 4.17, 4.21)

$$\begin{aligned}
\hat{H}\,\hat{a}^+\,\phi_E(x) &= (\,\hbar\omega\,\hat{a}^+\hat{a}^- \;+\; \frac{\hbar\omega}{2}\,\mathbb{1}\,)\,\hat{a}^+\,\phi_E(x)\\
&= \hat{a}^+\,(\,\hbar\omega\,\hat{a}^-\hat{a}^+ \;-\; \frac{\hbar\omega}{2}\,\mathbb{1} \;+\; \hbar\omega\,\mathbb{1}\,)\,\phi_E(x)\\
&= \hat{a}^+\,(\,\hat{H} \;+\; \hbar\omega\,)\,\phi_E(x)\\
&= (\,E \;+\; \hbar\omega\,)\,\hat{a}^+\,\phi_E(x)\;.
\end{aligned} \tag{4.23}$$

Together, it holds that for a stationary state $\phi_E(x)$ of energy E defined through (4.21) $\hat{a}^-\phi_E(x)$ is a stationary state of energy $E - \hbar\omega$ and $\hat{a}^+\phi_E(x)$ is a stationary state of energy $E + \hbar\omega$.
The results (4.22, 4.23) can be generalized to m-fold application of the operators $\hat{a}^+$ and $\hat{a}^-$

$$\begin{aligned}
\hat{H}\,(\hat{a}^+)^m\,\phi_E(x) &= (\,E \;+\; m\,\hbar\omega\,)\,\left(\hat{a}^+\right)^m\,\phi_E(x)\\
\hat{H}\,(\hat{a}^-)^m\,\phi_E(x) &= (\,E \;-\; m\,\hbar\omega\,)\,\left(\hat{a}^-\right)^m\,\phi_E(x)\;.
\end{aligned} \tag{4.24}$$

[1]This property states that the operators in the function space $\mathcal{N}_1$ are the hermitian conjugate of each other. This property of operators is investigated more systematically in Section 5.

One can use these relationships to construct an infinite number of stationary states by stepping up and down the spectrum in steps of $\pm\hbar\omega$. For obvious reasons one refers to $\hat{a}^+$ and $\hat{a}^-$ as ladder operators. Another name is creation ($\hat{a}^+$) and annihilation ($\hat{a}^-$) operators since these operators create and annihilate vibrational quanta $\hbar\omega$. There arise, however, two difficulties: (i) one needs to know at least one stationary state to start the construction; (ii) the construction appears to yield energy eigenvalues without lower bounds when, in fact, one expects that $E = 0$ should be a lower bound. It turns out that both difficulties can be resolved together. In fact, a state $\phi_0(x)$ which obeys the property

$$\hat{a}^- \phi_0(x) = 0 \tag{4.25}$$

on one side would lead to termination of the sequence $E_0 + m$, $m \in \mathbb{Z}$ when m is decreased, on the other side such a state is itself an eigenstate of $\hat{H}$ as can be shown using (4.17)

$$\hat{H}\,\phi_0(x) = (\,\hbar\omega\,\hat{a}^+\hat{a}^- + \frac{\hbar\omega}{2}\,\mathbb{1}\,)\,\phi_0(x) = \frac{1}{2}\hbar\omega\,\phi_0(x)\,. \tag{4.26}$$

Of course, the solution $\phi_0(x)$ of (4.25) needs to be normalizable in order to represent a bound state of the harmonic oscillator, i.e., $\phi_0(x)$ should be an element of the function space $\mathcal{N}_1$ defined in (4.5).

The property (**??**) has an important consequence for the stationary states $\phi_E(x)$. Let $\phi_E(x)$ and $\phi_{E'}(x)$ be two normalized stationary states corresponding to two *different* energies E, E', $E \neq E'$. For $f(x) = \phi_E(x)$ and $g(x) = \phi_{E'}(x)$ in (**??**) follows (Note that according to (**??**) the eigenvalue E is real.)

$$0 = \langle\phi_E|\hat{H}\phi_{E'}\rangle_{\Omega_\infty} - \overline{\langle\phi_{E'}|\hat{H}\phi_E\rangle_{\Omega_\infty}} = (\,E' - E\,)\,\langle\phi_E|\phi_{E'}\rangle_{\Omega_\infty}\,. \tag{4.27}$$

Since $E \neq E'$ one can conclude

$$\langle\phi_E|\phi_{E'}\rangle_{\Omega_\infty} = 0\,. \tag{4.28}$$

4.2 Ground State of the Harmonic Oscillator

A suitable solution of (4.25) can, in fact, be found. Using (4.6, 4.11) one can rewrite (4.25)

$$\left(\frac{d}{dy} + y\right)\phi_0(y) = 0 \tag{4.29}$$

where

$$y = \sqrt{\frac{m\omega}{\hbar}}\,x\,. \tag{4.30}$$

Assuming that $\phi_0(y)$ does not vanish anywhere in its domain $]-\infty, +\infty[$ one can write (4.29)

$$\frac{1}{\phi_0(y)}\frac{d}{dy}\phi_0(y) == \frac{d}{dy}\ln\phi_0(y) = -\,y\,, \tag{4.31}$$

the solution of which is

$$\ln\phi_0(y) = -\frac{1}{2}y^2 + c_0 \tag{4.32}$$

for some constant c_0 or

$$\phi_0(y) = c_0\exp\left(-\frac{1}{2}y^2\right)\,. \tag{4.33}$$

This solution is obviously normalizable. The conventional normalization condition

$$\langle \phi_0 | \phi_0 \rangle_{\Omega_\infty} \;=\; 1 \tag{4.34}$$

reads

$$\int_{-\infty}^{+\infty} dx\, |\phi_0(x)|^2 \;=\; |c_0|^2 \int_{-\infty}^{+\infty} dx \exp\left(-\frac{m\omega x^2}{2\hbar} \right) \;=\; |c_0|^2 \sqrt{\frac{\pi\hbar}{m\omega}}\;. \tag{4.35}$$

The appropriate ground state is

$$\phi_0(x) \;=\; \left[\frac{m\omega}{\pi\hbar} \right]^{\frac{1}{4}} \exp\left(-\frac{m\omega x^2}{2\hbar} \right)\;. \tag{4.36}$$

Since the first order differential equation (4.25) admits only one solution there is only one set of states with energy $E + m\,\hbar\omega$, $m \in \mathbb{Z}$ which properly terminate at some minimum value $E + m_0\,\hbar\omega \geq 0$. We recall that according to (4.26) the energy value associated with this state is $\frac{1}{2}\hbar\omega$. This state of lowest energy is called the *ground state* of the oscillator. The set of allowed energies of the oscillator according to (4.24) can then be written as follows

$$E_n \;=\; \hbar\omega\,(\,n \,+\, \frac{1}{2}\,)\,, \quad n \,=\, 0,\, 1,\, 2,\, \ldots \tag{4.37}$$

It is most remarkable that the energy $\frac{1}{2}\hbar\omega$ of the ground state is larger than the lowest classically allowed energy $E = 0$. The reason is that in the Hamiltonian (4.2) there are two competing contributions to the energy, the potential energy contribution which for a state $\delta(x)$, i.e., a state confined to the minimum corresponding to the classical state of lowest energy, would yield a vanishing contribution, and the kinetic energy contribution, which for a narrowly localized state yields a large positive value. The ground state (4.36) assumes a functional form such that both terms together assume a minimum value. We will consider this point more systematically in Section 5.

4.3 Excited States of the Harmonic Oscillator

Having obtained a suitable stationary state with lowest energy, we can now construct the stationary states corresponding to energies (4.37) above the ground state energy, i.e., we construct the states for $n = 1, 2, \ldots$, the so-called *excited states*. For this purpose we apply the operator $\hat{a}^+$ to the ground state (4.36) n times. Such states need to be suitably normalized for which purpose we introduce a normalization constant c'_n

$$\phi_n(x) \;=\; c'_n\, \left(\hat{a}^+\right)^n\, \phi_0(x)\,, \quad n \,=\, 0,\, 1,\, 2,\, \ldots \tag{4.38}$$

These states correspond to the energy eigenvalues (4.39), i.e., it holds

$$\hat{H}\,\phi_n(x) \;=\; \hbar\omega\,(\,n \,+\, \frac{1}{2}\,)\,\phi_n(x)\,, \quad n \,=\, 0,\, 1,\, 2,\, \ldots \tag{4.39}$$

We notice that the ground state wave function $\phi_0(x)$ as well as the operators $(\hat{a}^+)^n$ are real. We can, therefore, choose the normalization constants c'_n and the functions $\phi_n(x)$ real as well.

We need to determine now the normalization constants c'_n. To determine these constants we adopt a recursion scheme. For $n = 0$ holds $c'_0 = 1$. We consider then the situation that we have obtained a properly normalized state $\phi_n(x)$. A properly normalized state $\phi_{n+1}(x)$ is then of the form

$$\phi_{n+1}(x) \; = \; \alpha_n \, \hat{a}^+ \, \phi_n(x) \tag{4.40}$$

for some real constant α_n which is chosen to satisfy

$$\langle \phi_{n+1} | \phi_{n+1} \rangle_{\Omega_\infty} \; = \; \alpha_n^2 \, \langle \hat{a}^+ \phi_n | \hat{a}^+ \phi_n \rangle_{\Omega_\infty} \; = \; 1 \; . \tag{4.41}$$

Employing the adjoint property (4.20) yields

$$\alpha_n^2 \, \langle \phi_n | \hat{a}^- \hat{a}^+ \phi_n \rangle_{\Omega_\infty} \; = \; 1 \; . \tag{4.42}$$

Using (4.14) together with $\hat{H} \, \phi_n(x) \; = \; \hbar\omega(n + \frac{1}{2}) \, \phi_n(x)$ leads to the condition (note that we assumed $\phi_n(x)$ to be normalized)

$$\alpha_n^2 \, (\, n \, + \, 1 \,) \, \langle \phi_n | \hat{a}^- \hat{a}^+ \phi_n \rangle_{\Omega_\infty} \; = \; \alpha_n^2 \, (\, n \, + \, 1 \,) \; = \; 1 \tag{4.43}$$

From this follows $\alpha_n \; = \; 1/\sqrt{n+1}$ and, according to (4.40),

$$\hat{a}^+ \, \phi_n(x) \; = \; \sqrt{n+1} \, \phi_{n+1}(x) \; . \tag{4.44}$$

One can conclude then that the stationary states of the oscillator are described by the functions

$$\phi_n(x) \; = \; \frac{1}{\sqrt{n!}} \, \left(\hat{a}^+ \right)^n \, \phi_0(x) \; , \quad n \; = \; 0, 1, 2, \ldots \tag{4.45}$$

We like to note that these functions according to (4.28) and the construction (4.40–4.45) form an *orthonormal set*, i.e., they obey

$$\langle \phi_n | \phi_{n'} \rangle \; = \; \delta_{n \, n'} \; , \quad n, n' \; = \; 0, 1, 2, \ldots \; . \tag{4.46}$$

According to (4.24) holds in analogy to (4.40)

$$\phi_{n-1}(x) \; = \; \beta_n \, \hat{a}^- \, \phi_n(x) \tag{4.47}$$

for some suitable constants β_n. Since $\hat{a}^-$ is a real differential operator [see (4.18)] and since the $\phi_n(x)$ are real functions, β_n must be real as well. To determine β_n we note using (4.20)

$$1 \; = \; \langle \phi_{n-1} | \phi_{n-1} \rangle_{\Omega_\infty} \; = \; \beta_n^2 \, \langle \hat{a}^- \phi_n | \hat{a}^- \phi_n \rangle_{\Omega_\infty} \; = \; \beta_n^2 \, \langle \phi_n | \hat{a}^+ \hat{a}^- \phi_n \rangle_{\Omega_\infty} \; . \tag{4.48}$$

Equations (4.15 , 4.39) yield

$$1 \; = \; \beta_n^2 \, n \, \langle \phi_n | \phi_n \rangle_{\Omega_\infty} \; = \; \beta_n^2 \, n \; . \tag{4.49}$$

From this follows $\beta_n \; = \; 1/\sqrt{n}$ and, according to (4.47),

$$\hat{a}^- \, \phi_n(x) \; = \; \sqrt{n} \, \phi_{n-1}(x) \; . \tag{4.50}$$

Repeated application of this relationship yields

$$\phi_{n-s}(x) \; = \; \sqrt{\frac{(n-s)!}{n!}} \, \left(\hat{a}^- \right)^s \, \phi_n(x) \; . \tag{4.51}$$

Evaluating the Stationary States

We want to derive now an analytical expression for the stationary state wave functions $\phi_n(x)$ defined through (4.39). For this purpose we start from expression (4.45), simplifying the calculation, however, by introducing the variable y defined in (4.30) and employing the normalization

$$\int_{-\infty}^{+\infty} dy\, \phi_n^2(y) \;=\; 1 \tag{4.52}$$

This normalization of the wave functions differs from that postulated in (4.35) by a constant, n–independent factor, namely the square root of the Jacobian dx/dy, i.e., by

$$\sqrt{\left|\frac{dx}{dy}\right|} \;=\; \left[\frac{m\omega}{\hbar}\right]^{\frac{1}{4}} . \tag{4.53}$$

We will later re-introduce this factor to account for the proper normalization (4.35) rather than (4.52).
In terms of y the ground state wave function is

$$\phi_0(y) \;=\; \pi^{-\frac{1}{4}}\, e^{-\frac{y^2}{2}} \tag{4.54}$$

and $\hat{a}^+$ is

$$\hat{a}^+ \;=\; \frac{1}{\sqrt{2}} \left(y \;-\; \frac{d}{dy} \right) . \tag{4.55}$$

The eigenstates of the Hamiltonian are then given by

$$\phi_n(y) \;=\; \frac{1}{\sqrt{2^n\, n!\, \sqrt{\pi}}}\, e^{-\frac{y^2}{2}}\, e^{\frac{y^2}{2}} \left(y \;-\; \frac{d}{dy} \right)^n e^{-\frac{y^2}{2}} . \tag{4.56}$$

Relationship to Hermite Polynomials

We want to demonstrate now that the expression (4.56) can be expressed in terms of Hermite polynomials $H_n(y)$ introduced in Sect. 2.7 and given, for example, by the Rodrigues formula (2.200). We will demonstrate below the identity

$$H_n(y) \;=\; e^{\frac{y^2}{2}} \left(y \;-\; \frac{d}{dy} \right)^n e^{-\frac{y^2}{2}} \tag{4.57}$$

such that one can write the stationary state wave functions of the harmonic oscillator

$$\phi_n(y) \;=\; \frac{1}{\sqrt{2^n\, n!\, \sqrt{\pi}}}\, e^{-\frac{y^2}{2}}\, H_n(y) . \tag{4.58}$$

This result agrees with the expression (2.233) derived in Sect. 2.7. It should be noted that the normalization (4.28) of the ground state and the definition (4.57) which includes the factor $1/\sqrt{n!}$ according to Eqs. (4.40–4.46) yields a set of normalized states.

To verify the realtionship between the definition (4.57) of the *hermite* polynomials and the definition given by (2.200) we need to verify

$$\left(y \,-\, \frac{d}{dy} \right)^n e^{-\frac{y^2}{2}} \;=\; (-1)^n \, e^{\frac{y^2}{2}} \, \frac{d^n}{dy^n} \, e^{-y^2} \;. \tag{4.59}$$

which implies

$$H_n(y) \;=\; (-1)^n \, e^{y^2} \, \frac{d^n}{dy^n} \, e^{-y^2} \tag{4.60}$$

and, hence, the *Rodrigues formula* (2.200).
We prove (4.59) by induction noting first that (4.59) holds for $n = 0, 1$, and showing then that the property also holds for $n+1$ in case it holds for n, i.e.,

$$g(y) \;=\; (-1)^{n+1} \, e^{\frac{y^2}{2}} \, \frac{d^{n+1}}{dy^{n+1}} \, e^{-y^2} \;=\; \left(y \,-\, \frac{d}{dy} \right)^{n+1} e^{-\frac{y^2}{2}} \;. \tag{4.61}$$

One can factor $g(y)$ and employ (4.59) as follows

$$\begin{aligned} g(y) \;&=\; -e^{\frac{y^2}{2}} \, \frac{d}{dy} \, e^{-\frac{y^2}{2}} \, (-1)^n \, e^{\frac{y^2}{2}} \, \frac{d^n}{dy^n} \, e^{-y^2} \\ &=\; -e^{\frac{y^2}{2}} \, \frac{d}{dy} \, e^{-\frac{y^2}{2}} \left(y \,-\, \frac{d}{dy} \right)^n e^{-\frac{y^2}{2}} \;. \end{aligned} \tag{4.62}$$

Denoting $f(y) = (y - d/dy)\exp(-y^2/2)$ and employing

$$\frac{d}{dy} \, e^{-\frac{y^2}{2}} \, f(y) \;=\; -y \, e^{-\frac{y^2}{2}} \, f(y) \,+\, e^{-\frac{y^2}{2}} \, \frac{d}{dy} \, f(y) \tag{4.63}$$

one obtains

$$g(y) \;=\; \left(y \,-\, \frac{d}{dy} \right) f(y) \tag{4.64}$$

which implies that (4.61) and, therefore, (4.59) hold.

4.4 Propagator for the Harmonic Oscillator

We consider now the solution of the time-dependent Schrödinger equation of the harmonic oscillator (4.1, 4.2) for an arbitrary initial wave function $\psi(x_0, t_0)$. Our derivation will follow closely the procedure adopted for the case of a 'particle in a box' [see Eqs. (3.106–3.114)]. For the sake of notational simplicity we employ initially the coordinate y as defined in (4.30) and return later to the coordinate x.
Starting point of our derivation is the assumption that the initial condition can be expanded in terms of the eigenstates $\phi_n(y)$ (4.39)

$$\psi(y_0, t_0) \;=\; \sum_{n=0}^{\infty} d_n \, \phi_n(y_0) \;. \tag{4.65}$$

Such expansion is possible for any $f(y_0) = \psi(y_0, t_0)$ which is an element of $\mathcal{N}_1$ defined in (4.5), a supposition which is not proven here[2]. Employing orthogonality condition (4.46) the expansion coefficients d_n are

$$d_n = \int_{-\infty}^{+\infty} dy_0 \, \psi(y_0, t_0) \, \phi_n(y_0) \, . \tag{4.66}$$

One can extend expansion (4.65) to times $t \geq t_0$ through insertion of time-dependent coefficients $c_n(t_1)$

$$\psi(y, t_1) = \sum_{n=0}^{\infty} d_n \, \phi_n(y) \, c_n(t) \tag{4.67}$$

for which according to (3.108–3.112) and (4.39) holds

$$c_n(t_1) = \exp\left(-i \, \omega \, (\, n + \frac{1}{2} \,) \, (\, t_1 - t_0 \,) \right) \, . \tag{4.68}$$

Altogether one can express then the solution

$$\psi(y, t_1) = \int_{-\infty}^{+\infty} dy_0 \, \phi(y, t | y_0, t_0) \, \psi(y_0, t) \tag{4.69}$$

where

$$\phi(y, t_1 | y_0, t_0) = \sum_{n=0}^{\infty} \phi_n(y) \, \phi_n(y_0) \, t^{n + \frac{1}{2}} \, , \quad t = e^{-i\omega(t_1 - t_0)} \tag{4.70}$$

is the propagator of the linear harmonic oscillator. We want to demonstrate now that this propagator is identical to the propagator (2.147) for the harmonic iscillator determined in Sect. sec:harm. In order to prove the equivalence of (4.70) and (2.147) we employ the technique of generating functions as in Sect. 2.7. For this purpose we start from the integral representation (2.225) which allows one to derive a generating function for products of *Hermite polynomials* which can be applied to the r.h.s. of (4.70). We consider for this purpose the following expression for $|t| < 1$

$$w(y, y_0, t) = \sum_{n=0}^{\infty} \frac{H_n(y) e^{-y^2/2} \, H_n(y_0) e^{-y_0^2/2}}{2^n \, n! \, \sqrt{\pi}} \, t^n \, . \tag{4.71}$$

Applying (**??**) to express $H_n(y)$ and $H_n(y_0)$ yields

$$\begin{aligned} w(y, y_0, t) = \; & \pi^{-3/2} \exp\left(\frac{y^2}{2} + \frac{y_0^2}{2} \right) \int_{-\infty}^{+\infty} du \int_{-\infty}^{+\infty} dv \overbrace{\sum_{n=0}^{\infty} \frac{1}{n!} \, (\, -2tuv \,)^n}^{\exp(-2tuv)} \times \\ & \times \exp(\, -u^2 - v^2 + 2iyu + 2iy_0 v \,) \, . \end{aligned} \tag{4.72}$$

[2]A demonstration of this property can be found in *Special Functions and their Applications* by N.N.Lebedev (Prentice Hall, Inc., Englewood Cliffs, N.J., 1965) Sect. 4.15, pp. 68; this is an excellent textbook from which we have borrowed heavily in this Section.

Carrying out the sum on the r.h.s. one obtains

$$w(y, y_0, t) = \pi^{-3/2} \exp\left(\frac{y^2}{2} + \frac{y_0^2}{2}\right) \int_{-\infty}^{+\infty} dv \exp(-v^2 + 2iy_0 v) \times \times \underbrace{\int_{-\infty}^{+\infty} du \exp(-u^2 - 2u(tv - iy))}_{\sqrt{\pi}\exp(t^2v^2 - 2iytv - y^2)} . \tag{4.73}$$

The incomplete Gaussian integral

$$\int_{-\infty}^{+\infty} dx\, e^{-a^2x^2 - 2bx} = \frac{\sqrt{\pi}}{a} e^{b^2/a^2} , \quad \mathrm{Re}\, a^2 > 0 \tag{4.74}$$

applied once results in

$$w(y, y_0, t) = \pi^{-1} \exp\left(-\frac{y^2}{2} + \frac{y_0^2}{2}\right) \int_{-\infty}^{+\infty} dv \exp(-(1 - t^2)\, v^2 - 2i\,(yt - y_0)\, v) . \tag{4.75}$$

Applying (4.74) a second time yields finally together with the definition (4.71) of $w(y, y_0, t)$

$$\frac{1}{\sqrt{\pi(1-t^2)}} \exp\left[-\tfrac{1}{2}(y^2 + y_0^2)\tfrac{1+t^2}{1-t^2} + 2\,y\,y_0\,\tfrac{t}{1-t^2}\right] = \textstyle\sum_{n=0}^{\infty} \frac{H_n(y)e^{-y^2/2}\,H_n(y_0)e^{-y_0^2/2}}{2^n\, n!\,\sqrt{\pi}}\, t^n . \tag{4.76}$$

One can express this in terms of the stationary states (4.58) of the harmonic oscillator

$$\sum_{n=0}^{\infty} \phi_n(y)\,\phi_n(y_0)\, t^n = \frac{1}{\sqrt{\pi(1-t^2)}} \exp\left[-\frac{1}{2}(y^2 + y_0^2)\frac{1 + t^2}{1 - t^2} + 2\,y\,y_0 \frac{t}{1 - t^2}\right] . \tag{4.77}$$

The sum in (4.70) is, indeed, identical to the generating function (4.77), i.e., it holds

$$\phi(y, t_1|y_0, t_0) = \frac{1}{\sqrt{2\,i\,\pi\,F(t_1, t_0)}} \exp\left[\frac{i}{2}(y^2 + y_0^2)\,G(t_1, t_0) - i\,y\,y_0 \frac{1}{F(t_1, t_0)}\right] \tag{4.78}$$

where

$$F(t_1, t_0) = \frac{1 - e^{-2i\omega(t_1 - t_0)}}{2\,i\,e^{-i\omega(t_1 - t_0)}} = \sin\omega(t_1 - t_0) \tag{4.79}$$

and

$$G(t_1, t_0) = i\,\frac{1 + e^{-2i\omega(t_1 - t_0)}}{1 - e^{-2i\omega(t_1 - t_0)}} = \frac{\cos\omega(t_1 - t_0)}{\sin\omega(t_1 - t_0)} . \tag{4.80}$$

We can finally express the propagator (4.78) in terms of the coordinates x and x_0. This requires that we employ (4.30) to replace y and y_0 and that we multiply the propagator by $\sqrt{m\omega/\hbar}$, i.e., by a factor $\sqrt[4]{m\omega/\hbar}$ for both $\phi_n(y)$ and $\phi_n(y_0)$. The resulting propagator is

$$\phi(x, t|x_0, t_0) = \left[\frac{m\omega}{2i\pi\hbar\sin\omega(t_1 - t_0)}\right]^{\frac{1}{2}} \times \exp\left\{\frac{im\omega}{2\hbar\sin\omega(t_1 - t_0)}\left[(x^2 + x_0^2)\cos\omega(t_1 - t_0) - 2\,x\,x_0\right]\right\} . \tag{4.81}$$

This result agrees with the propagator (2.147) derived by means of the path integral description.

4.5 Working with Ladder Operators

In the last section we have demonstrated the use of differential equation techniques, the use of generating functions. We want to introduce now techniques based on the ladder operators $\hat{a}^+$ and $\hat{a}^-$. For the present there is actually no pressing need to apply such techniques since the techniques borrowed from the theory of differential equations serve us well in describing harmonic oscillator type quantum systems. The reason for introducing the calculus of the operators $\hat{a}^+$ and $\hat{a}^-$ is that this calculus proves to be useful for the description of vibrations in crystals, i.e., phonons, and of the modes of the quantized electromagnetic field; both quantum systems are endowed with a large number of modes, each corresponding to a single harmonic oscillator of the type studied presently by us. It is with quantum electrodynamics and solid state physics in mind that we cease the opportunity of the single quantum mechanical harmonic oscillator to develop a working knowledge for $\hat{a}^+$ and $\hat{a}^-$ in the most simple setting.
To put the following material in the proper modest perspective we may phrase it as an approach which rather than employing the coordinate y and the differential operator d/dy uses the operators

$$\hat{a}^+ = \frac{1}{\sqrt{2}}\left(y - \frac{d}{dy}\right), \quad \hat{a}^- = \frac{1}{\sqrt{2}}\left(y + \frac{d}{dy}\right). \tag{4.82}$$

Obviously, one can express $y = \sqrt{2}(\hat{a}^- + \hat{a}^+)$ and $d/dy = \sqrt{2}(\hat{a}^- - \hat{a}^+)$ and, hence, the approaches using y, d/dy and $\hat{a}^+$, $\hat{a}^-$ must be equivalent.

Calculus of Creation and Annihilation Operators

We summarize first the key properties of the operators $\hat{a}^+$ and $\hat{a}^-$

$$\begin{aligned}
[\hat{a}^-, \hat{a}^+] &= 1\!\!1 \\
\hat{a}^- \, \phi_0(y) &= 0 \\
\hat{a}^+ \, \phi_n(y) &= \sqrt{n+1}\, \phi_{n+1}(y) \\
\hat{a}^- \, \phi_n(y) &= \sqrt{n}\, \phi_{n-1}(y) \\
\langle \hat{a}^- f | g \rangle_{\Omega_\infty} &= \langle f | \hat{a}^+ g \rangle_{\Omega_\infty} .
\end{aligned} \tag{4.83}$$

We note that these properties imply

$$\phi_n(y) = (\hat{a}^+)^n \phi_0(y)/\sqrt{n!}\,. \tag{4.84}$$

We will encounter below functions of $\hat{a}^+$ and $\hat{a}^-$, e.g., $f(\hat{a}^+)$. Such functions are defined for $f : \mathbb{R} \to \mathbb{R}$ in case that the Taylor expansion

$$f(y) = \sum_{\nu=0}^{\infty} \frac{f^{(\nu)}(0)}{\nu!}\, y^\nu \tag{4.85}$$

is convergent everywhere in $\mathbb{R}$. Here $f^{(\nu)}(y_0)$ denotes the ν-th derivative of $f(y)$ taken at $y = y_0$. In this case we define

$$f(\hat{a}^+) = \sum_{\nu=0}^{\infty} \frac{f^{(\nu)}(0)}{\nu!}\, \left(\hat{a}^+\right)^\nu \tag{4.86}$$

and similarly for $f(\hat{a}^-)$. The following important property holds

$$\hat{a}^- f(\hat{a}^+) \;=\; f^{(1)}(\hat{a}^+) \;+\; f(\hat{a}^+)\,\hat{a}^- \,. \tag{4.87}$$

In particular,

$$\hat{a}^- f(\hat{a}^+)\,\phi_0(y) \;=\; f^{(1)}(\hat{a}^+)\,\phi_0(y) \tag{4.88}$$

which follows from $\hat{a}^- \phi_o(y) = 0$. We note

$$\hat{a}^- f(\hat{a}^+) \;=\; [\hat{a}^-, f(\hat{a}^+)] \;+\; f(\hat{a}^+)\,\hat{a}^- \tag{4.89}$$

which implies that in order to prove (4.87, 4.88) we need to show actually

$$[\hat{a}^-, f(\hat{a}^+)] \;=\; f^{(1)}(\hat{a}^+)\,. \tag{4.90}$$

To prove (4.90) we show that (4.90) holds for any function $f_n(\hat{a}^+) = (\hat{a}^+)^n$ which is a power of $\hat{a}^+$. The convergence of the Taylor expansion ascertains then that (4.90) holds for $f(\hat{a}^+)$.
We proceed by induction noticing first that (4.90) holds for f_0 and for f_1. The first case is trivial, the second case follows from

$$[\hat{a}^-, f_1(\hat{a}^+)] \;=\; [\hat{a}^-, \hat{a}^+] \;=\; 1\!\!1 \;=\; f_1^{(1)}(\hat{a}^+)\,. \tag{4.91}$$

Let us assume that (4.90) holds for f_n. For f_{n+1} follows then

$$\begin{aligned} [\hat{a}^-, f_{n+1}(\hat{a}^+)] \;&=\; [\hat{a}^-, (\hat{a}^+)^n \hat{a}^+] \;=\; (\hat{a}^+)^n\,[\hat{a}^-, \hat{a}^+] \;+\; [\hat{a}^-, (\hat{a}^+)^n]\,\hat{a}^+ \\ &=\; (\hat{a}^+)^n\,1\!\!1 \;+\; n\,(\hat{a}^+)^{n-1}\hat{a}^+ \;=\; (n+1)\,(\hat{a}^+)^n \end{aligned} \tag{4.92}$$

Since any function $f(y)$ in the proper function space $\mathcal{N}_1$ can be expanded

$$f(y) \;=\; \sum_{n=0}^{\infty} d_n\,\phi_n(y) \;=\; \sum_{n=0}^{\infty} d_n\,\frac{(\hat{a}^+)^n}{\sqrt{n!}}\,\phi_0(y) \tag{4.93}$$

one can reduce all operators acting on some proper state function by operators acting on the state $\phi_0(y)$. Hence, property (4.88) is a fundamental one and will be used in the following, i.e., we will only assume operator functions acting on $\phi_0(y)$. As long as the operators act on $\phi_0(y)$ one can state then that $\hat{a}^-$ behaves like a differential operator with respect to functions $f(\hat{a}^+)$. Note that an immediate consequence of (4.88) is

$$(\hat{a}^-)^n\, f(\hat{a}^+)\,\phi_o(y) \;=\; f^{(n)}(\hat{a}^+)\,\phi_0(y)\,. \tag{4.94}$$

We like to state the following property of the functions $f(\hat{a}^\pm)$

$$\langle f(\hat{a}^-)\,\phi|\psi\rangle \;=\; \langle\phi|f(\hat{a}^+)\,\psi\rangle\,. \tag{4.95}$$

This identity follows from (4.83) and can be proven for all powers f_n by induction and then inferred for all proper functions $f(\hat{a}^\pm)$.
An important operator function is the exponential function. An example is the so-called shift operator $\exp(u\hat{a}^-)$. It holds

$$e^{u\hat{a}^-}\, f(\hat{a}^+)\,\phi_0(y) \;=\; f(\hat{a}^+ \,+\, u)\,\phi_0(y)\,. \tag{4.96}$$

To prove this property we expand $\exp(u\hat{a}^-)$

$$\sum_{\nu=0}^{\infty} \frac{u^\nu}{\nu!}\,(\hat{a}^-)^\nu\, f(\hat{a}^+)\,\phi_0(y) \;=\; \sum_{\nu=0}^{\infty} \frac{u^\nu}{\nu!}\, f^{(\nu)}(\hat{a}^+)\,\phi_0(y) \;=\; f(\hat{a}^+ + u)\,\phi_0(y)\;. \tag{4.97}$$

An example in which (4.96) is applied is

$$e^{u\hat{a}^-}\, e^{v\hat{a}^+}\,\phi_0(y) \;=\; e^{v\,(\hat{a}^+ + u)}\,\phi_0(y) \;=\; e^{uv}\, e^{v\hat{a}^+}\,\phi_0(y)\;. \tag{4.98}$$

The related operators $\exp[v\hat{a}^+ \pm v^*\hat{a}^-]$ play also an important role. *We assume in the following derivation, without explicitly stating this, that the operators considered act on $\phi_0(y)$.* To express these operators as products of operators $\exp(v\hat{a}^+)$ and $\exp(v^*\hat{a}^-)$ we consider the operator $\hat{C}(u) = \exp(uz\hat{a}^+)\exp(uz^*\hat{a}^-)$ where $u \in \mathbb{R}$, $z \in \mathbb{C}$, $\digamma\digamma^* = \nVdash$ and determine its derivative

$$\begin{aligned} \tfrac{d}{du}\,\hat{C}(u) \;&=\; z\hat{a}^+ e^{uz\hat{a}^+}\, e^{uz^*\hat{a}^-} \;+\; e^{uz\hat{a}^+}\, z^*\hat{a}^-\, e^{uz^*\hat{a}^-} \;=\; \\ &\left(\, z\hat{a}^+ + z^*\hat{a}^- \,\right)\, e^{uz\hat{a}^+}\, e^{uz^*\hat{a}^-} \;-\; \left[z^*\hat{a}^-, e^{uz\hat{a}^+} \right]\, e^{uz^*\hat{a}^-} \end{aligned} \tag{4.99}$$

Using (4.90) and $zz^* = 1$ we can write this

$$\begin{aligned} \tfrac{d}{du}\,\hat{C}(u) \;&=\; \left(\, z\hat{a}^+ + z^*\hat{a}^- \,\right)\, e^{uz\hat{a}^+}\, e^{uz^*\hat{a}^-} \;-\; u\, e^{uz\hat{a}^+}\, e^{uz^*\hat{a}^-} \;=\; \\ &\left(\, z\hat{a}^+ + z^*\hat{a}^- - u \,\right)\, \hat{C}(u)\;. \end{aligned} \tag{4.100}$$

To solve this differential equation we define $\hat{C}(u) = \hat{D}(u)\exp(-u^2/2)$ which leads to

$$\frac{d}{du}\,\hat{D}(u) \;=\; \left(\, z\hat{a}^+ + z^*\hat{a}^- \,\right)\, \hat{D}(u) \tag{4.101}$$

$\hat{C}(u)$ obviously obeys $\hat{C}(0) = \mathbb{1}$. This results in $\hat{D}(0) = \mathbb{1}$ and, hence, the solution of (4.101) is $\hat{D}(u) = \exp(uz\hat{a}^+ + uz^*\hat{a}^-)$. One can conclude then using the definition of $\hat{C}(u)$ and defining $v = uz$

$$e^{v\hat{a}^+ + v^*\hat{a}^-}\,\phi_0(y) \;=\; e^{\frac{1}{2}vv^*}\, e^{v\hat{a}^+}\, e^{v^*\hat{a}^-}\,\phi_0(y)\;. \tag{4.102}$$

Similarly, one obtains

$$e^{v\hat{a}^+ - v^*\hat{a}^-}\,\phi_0(y) \;=\; e^{-\frac{1}{2}vv^*}\, e^{v\hat{a}^+}\, e^{-v^*\hat{a}^-}\,\phi_0(y) \tag{4.103}$$

$$e^{v\hat{a}^+ + v^*\hat{a}^-}\,\phi_0(y) \;=\; e^{-\frac{1}{2}vv^*}\, e^{v^*\hat{a}^-}\, e^{v\hat{a}^+}\,\phi_0(y) \tag{4.104}$$

$$e^{v\hat{a}^+ - v^*\hat{a}^-}\,\phi_0(y) \;=\; e^{\frac{1}{2}vv^*}\, e^{-v^*\hat{a}^-}\, e^{v\hat{a}^+}\,\phi_0(y)\;. \tag{4.105}$$

Below we will use the operator identity which follows for the choice of $v = \alpha$ in (4.105)

$$e^{\alpha\hat{a}^+ - \alpha^*\hat{a}^-}\,\phi_0(y) \;=\; e^{\frac{\alpha\alpha^*}{2}}\, e^{-\alpha^*\hat{a}^-}\, e^{\alpha\hat{a}^+}\,\phi_0(y)\;. \tag{4.106}$$

Applying (4.98) for $u = -\alpha^*$ and $v = \alpha$ yields

$$e^{-\alpha^*\hat{a}^-}\, e^{\alpha\hat{a}^+}\,\phi_0(y) \;=\; e^{-\alpha\alpha^*}\, e^{\alpha\hat{a}^+}\,\phi_0(y)\;. \tag{4.107}$$

This together with (4.106) yields

$$e^{\alpha\hat{a}^+ - \alpha^*\hat{a}^-}\,\phi_0(y) \;=\; e^{-\frac{\alpha\alpha^*}{2}}\, e^{\alpha\hat{a}^+}\,\phi_0(y)\;. \tag{4.108}$$

Generating Function in Terms of $\hat{a}^+$

We want to demonstrate now that generating functions are as useful a tool in the calculus of the ladder operators as they are in the calculus of differential operators. We will derive the equivalent of a generating function and use it to rederive the values $H_n(0)$ and the orthonormality properties of $\phi_n(y)$.
We start from the generating function (**??**) and replace according to (4.58) the *Hermite* polynomials $H_n(y)$ by the eigenstates $\phi_n(y)$

$$e^{2yt - t^2} \;=\; \pi^{\frac{1}{4}} \sum_{n=0}^{\infty} \frac{(\sqrt{2}t)^n}{\sqrt{n!}} \, e^{y^2/2} \, \phi_n(y) \tag{4.109}$$

Using (4.84) and defining $\sqrt{2}t \;=\; u$ one can write then

$$\pi^{-\frac{1}{4}} \exp\left(-\frac{y^2}{2} \;+\; \sqrt{2}\, u\, y \;-\; \frac{u^2}{2} \right) \;=\; \sum_{n=0}^{\infty} \frac{u^n}{n!} \, (\hat{a}^+)^n \, \phi_0(y) \tag{4.110}$$

or

$$\pi^{-\frac{1}{4}} \exp\left(-\frac{y^2}{2} \;+\; \sqrt{2}\, u\, y \;-\; \frac{u^2}{2} \right) \;=\; e^{u\hat{a}^+} \, \phi_0(y) \; . \tag{4.111}$$

This expression, in the ladder operator calculus, is the equivalent of the generating function (**??**). We want to derive now the values $H_n(0)$. Setting $y \;=\; 0$ in (4.111) yields

$$\pi^{-\frac{1}{4}} \exp\left(-\frac{u^2}{2} \right) \;=\; e^{u\hat{a}^+} \, \phi_0(0) \; . \tag{4.112}$$

Expanding both sides of this equation and using (4.84, 4.58) one obtains

$$\pi^{-\frac{1}{4}} \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu} u^{2\nu}}{2^{\nu} \nu!} \;=\; \sum_{\nu=0}^{\infty} \frac{u^{\nu}}{\sqrt{\nu!}} \, \phi_{\nu}(0) \;=\; \sum_{\nu=0}^{\infty} \frac{u^{\nu}}{\sqrt{2^{\nu} \sqrt{\pi}\, \nu!}} \, H_{\nu}(0) \tag{4.113}$$

Comparision of all terms on the l.h.s. and on the r.h.s. provide the same values for $H_n(0)$ as provided in Eq. (**??**).
Similarly, we can reproduce by means of the generating function (4.111) the orthonormality properties of the wave functions $\phi_n(y)$. For this purpose we consider

$$\langle e^{u\hat{a}^+} \, \phi_0 | e^{v\hat{a}^+} \, \phi_0 \rangle \;=\; \langle \phi_0 | e^{u\hat{a}^-} \, e^{v\hat{a}^+} \, \phi_0 \rangle \tag{4.114}$$

where we have employed (4.95). Using (4.98) and again (4.95) one obtains

$$\langle e^{u\hat{a}^+} \, \phi_0 | e^{v\hat{a}^+} \, \phi_0 \rangle \;=\; \langle \phi_0 | e^{uv} \, e^{v\hat{a}^+} \, \phi_0 \rangle \;=\; e^{uv} \, \langle e^{v\hat{a}^-} \, \phi_0 | \phi_0 \rangle \;=\; e^{uv} \tag{4.115}$$

where the latter step follows after expansion of $e^{v\hat{a}^-}$ and using $\hat{a}^- \, \phi_0(y) \;=\; 0$. Expanding r.h.s. and l.h.s. of (4.115) and using (4.84) yields

$$\sum_{\nu,\mu=0}^{\infty} \frac{u^{\mu} \, v^{\nu}}{\sqrt{\mu! \nu!}} \, \langle \phi_{\mu} | \phi_{\nu} \rangle \;=\; \sum_{\mu=0}^{\infty} \frac{u^{\mu} \, v^{\mu}}{\mu!} \tag{4.116}$$

from which follows by comparision of each term on both sides $\langle \phi_{\mu} | \phi_{\nu} \rangle \;=\; \delta_{\mu\,\nu}$, i.e., the expected orthonormality property.

4.6 Momentum Representation for the Harmonic Oscillator

The description of the harmonic oscillator provided so far allows one to determine the probability density $P(x)$ of finding an oscillator at position x. For example, for an oscillator in a stationary state $\phi_n(x)$ as given by (2.235), the probability density is

$$P(x) \;=\; |\phi_n(x)|^2 \,. \tag{4.117}$$

In this section we want to provide a representation for the harmonic oscillator which is most natural if one wishes to determine for a stationary state of the system the probability density $\tilde{P}(p)$ of finding the system at momentum p. For this purpose we employ the Schrödinger equation in the momentum representation.
In the position representation the wave function is a function of x, i.e., is given by a function $\psi(x)$; the momentum and position operators are as stated in (4.6) and the Hamiltonian (4.7) is given by (4.2). In the momentum representation the wave function is a function of p, i.e., is given by the function $\tilde{\psi}(p)$; the momentum and position operators are

$$\hat{p} \;=\; p \,, \quad \hat{x} \;=\; i\hbar\frac{d}{dp} \,, \tag{4.118}$$

and the Hamiltonian (4.7) is

$$\hat{H} \;=\; \frac{1}{2m}\,p^2 \;-\; \frac{m\hbar^2\omega^2}{2}\,\frac{d^2}{dp^2} \,. \tag{4.119}$$

Accordingly, the time-independent Schrödinger equation for the oscillator can be written

$$\left(-\frac{m\omega^2\hbar^2}{2}\,\frac{d^2}{dp^2} \;+\; \frac{1}{2m}\,p^2 \right) \tilde{\phi}_E(p) \;=\; E\,\tilde{\phi}_E(p) \,. \tag{4.120}$$

Multiplying this equation by $1/m^2\omega^2$ yields

$$\left(-\frac{\hbar^2}{2m}\,\frac{d^2}{dp^2} \;+\; \frac{1}{2}m\tilde{\omega}^2\,p^2 \right) \tilde{\phi}_E(p) \;=\; \tilde{E}\,\tilde{\phi}_E(p) \tag{4.121}$$

where

$$\tilde{E} \;=\; E\,/\,m^2\omega^2 \,, \qquad \tilde{\omega} \;=\; 1\,/\,m^2\omega \,. \tag{4.122}$$

For a solution of the Schrödinger equation in the momentum representation, i.e., of (4.121), we note that the posed eigenvalue problem is formally identical to the Schrödinger equation in the position representation as given by (4.2, 4.3). The solutions can be stated readily exploiting the earlier results. The eigenvalues, according to (4.37), are

$$\tilde{E}_n \;=\; \hbar\tilde{\omega}\,(n \,+\, \tfrac{1}{2}) \,, \quad n \;=\; 0,1,2,\ldots \tag{4.123}$$

or, using (4.122),

$$E_n \;=\; \hbar\omega\,(n \,+\, \tfrac{1}{2})) \,, \quad n \;=\; 0,1,2,\ldots \tag{4.124}$$

As to be expected, the eigenvalues are identical to those determined in the position representation. The momentum representation eigenfunctions are, using (2.235),

$$\tilde{\phi}_n(p) \;=\; \frac{(-i)^n}{\sqrt{2^n\,n!}}\,\left[\frac{m\tilde{\omega}}{\pi\hbar}\right]^{\frac{1}{4}} \exp\left(-\frac{m\tilde{\omega}p^2}{2\hbar}\right)\,H_n(\sqrt{\frac{m\tilde{\omega}}{\hbar}}\;p) \,. \tag{4.125}$$

or with (4.122)

$$\tilde{\phi}_n(p) \;=\; \frac{(-i)^n}{\sqrt{2^n\, n!}} \left[\frac{1}{\pi m\hbar\omega}\right]^{\frac{1}{4}} \exp\left(-\frac{p^2}{2m\hbar\omega}\right) H_n\left(\sqrt{\frac{1}{m\hbar\omega}}\; p\right) . \tag{4.126}$$

Normalized eigenfunctions are, of course, defined only up to an arbitrary phase factor $e^{i\alpha}$, $\alpha \in \mathbb{R}$; this allowed us to introduce in (4.125, 4.126) a phase factor $(-i)^n$.
We can now state the probability density $\tilde{P}_n(p)$ for an oscillator with energy E_n to assume momentum p. According to the theory of the momentum representation holds

$$\tilde{P}_n(p) \;=\; |\tilde{\phi}_n(p)|^2 \tag{4.127}$$

or

$$\tilde{P}_n(p) \;=\; \frac{1}{2^n\, n!} \left[\frac{1}{\pi m\hbar\omega}\right]^{\frac{1}{2}} \exp\left(-\frac{p^2}{m\hbar\omega}\right) H_n^2\left(\sqrt{\frac{1}{m\hbar\omega}}\; p\right) . \tag{4.128}$$

We may also express the eigenstates (4.126) in terms of dimensionless units following the procedure adopted for the position representation where we employed the substitution (4.30) to obtain (4.58). Defining $k \;=\; \sqrt{m\tilde{\omega}/\hbar}\, p$ or

$$k \;=\; \sqrt{\frac{1}{m\hbar\omega}}\; p \tag{4.129}$$

one obtains, instead of (4.126),

$$\phi_n(k) \;=\; \frac{(-i)^n}{\sqrt{2^n\, n!\, \sqrt{\pi}}}\, e^{-\frac{k^2}{2}}\, H_n(k) . \tag{4.130}$$

We want to finally apply the relationship

$$\tilde{\phi}_n(p) \;=\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx\, \exp(-ipx/\hbar)\, \phi_n(x) \tag{4.131}$$

in the present case employing dimensionless units. From (4.30) and (4.129) follows

$$k\, y \;=\; p\, x\, /\, \hbar \tag{4.132}$$

and, hence,

$$\tilde{\phi}_n(k) \;=\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dy\, \exp(-iky)\, \phi_n(y) \tag{4.133}$$

where we note that a change of the normalization of $\phi_n(y)$ [c.f. (4.30) and (2.235)] absorbs the replacement $dx \rightarrow dy$, i.e., the Jacobian. This implies the property of Hermite polynomials

$$(-i)^n\, e^{-\frac{k^2}{2}}\, H_n(k) \;=\; \int_{-\infty}^{+\infty} dy\, \exp(-iky)\, e^{-\frac{y^2}{2}}\, H_n(y) \tag{4.134}$$

Exercise 4.6.1: Demonstrate the validity of (4.134), i.e., the correctness of the phase factors $(-i)^n$, through direct integration. Proceed as follows.

(a) Prove first the property

$$f(k,z) \;=\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dy \, \exp(-iky)\, g(z,y) \tag{4.135}$$

where

$$f(k,z) \;=\; \exp\left(-2ikz \,+\, z^2 \,-\, k^2/2\right) \;, \qquad g(k,z) \;=\; \exp\left(2yz \,-\, z^2 \,-\, y^2/2\right) \tag{4.136}$$

(b) Employ the generating function (2.194, 2.196) of Hermite polynomials and equate coefficients of equal powers of z to prove (4.134).

4.7 Quasi-Classical States of the Harmonic Oscillator

A classical harmonic oscillator, for example, a pendulum swinging at small amplitudes, carries out a periodic motion described by

$$y(t) \;=\; \mathrm{Re}\left(\, y_0 \, e^{i\omega t}\,\right) \tag{4.137}$$

where $y(t)$ describes the center of the particle and where the mass of the particle remains narrowly distributed around $y(t)$ for an arbitrary period of time. The question arises if similar states exist also for the quantum oscillator. The answer is 'yes'. Such states are referred to as *quasi-classical states*, *coherent states*, or *Glauber states* of the harmonic oscillator and they play, for example, a useful role in Quantum Electrodynamics since they allow to reproduce with a quantized field as closely as possible the properties of a classical field[3]. We want to construct and characterize such states.

The *quasi-classical states* can be obtained by generalizing the construction of the ground state of the oscillator, namely through the eigenvalue problem for normalized states

$$\hat{a}^- \,\phi^{(\alpha)}(y) \;=\; \alpha\, \phi^{(\alpha)}(y) \;, \quad \alpha \,\in\, \mathbb{R} \;, \quad \langle \phi^{(\alpha)} \,|\, \phi^{(\alpha)} \rangle \;=\; \nVdash \;. \tag{4.138}$$

We will show that a harmonic oscillator prepared in such a state will remain forever in such a state, except that α changes periodically in time. We will find that α can be characterized as a displacement form the minimum of the oscillator and that the state $\phi^{(\alpha)}(y)$ displays the same spatial probability distribution as the ground state. The motion of the probability distribution of such state is presented in Fig. 4.1 showing in its top row the attributes of such state just discussed. We first construct the solution of (4.138). For this purpose we assume such state exists and expand

$$\phi^{(\alpha)}(y) \;=\; \sum_{n=0}^{\infty} d_n^{(\alpha)}\, \phi_n(y) \;, \quad \sum_{n=0}^{\infty} \left| d_n^{(\alpha)} \right|^2 \;=\; 1 \tag{4.139}$$

where we have added the condition that the states be normalized. Inserting this expansion into (4.138) yields using $\hat{a}^- \,\phi_n(y) \;=\; \sqrt{n}\, \phi_{n-1}(y)$

$$\sum_{n=0}^{\infty} \left(\sqrt{n}\, d_n^{(\alpha)}\, \phi_{n-1}(y) \;-\; \alpha\, d_n^{(\alpha)}\, \phi_n(y) \right) \;=\; 0 \tag{4.140}$$

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[3]An excellent textbook is *Photons and Atoms: Introduction to Quantum Electrodynamics* by C.Chen-Tannoudji, J.Dupont-Roc, and G.Grynberg (John Wiley & Sons, Inc., New York, 1989) which discusses quasi-classical states of the free electromagnetic field in Sect.— III.C.4, pp. 192

or

$$\sum_{n=0}^{\infty} \left(\sqrt{n+1}\, d_{n+1}^{(\alpha)} \;-\; \alpha\, d_n^{(\alpha)} \right) \phi_n(y) \;=\; 0 \,. \tag{4.141}$$

Because of the linear independence of the states $\phi_n(y)$ all coefficients multiplying $\phi_n(y)$ must vanish and on obtains the recursion relationship

$$d_{n+1}^{(\alpha)} \;=\; \frac{\alpha}{\sqrt{n+1}}\, d_n^{(\alpha)} \,. \tag{4.142}$$

The solution is

$$d_n^{(\alpha)} \;=\; \frac{\alpha^n}{\sqrt{n!}}\, c \tag{4.143}$$

for a constant c which is determined through the normalization condition [see (4.139)]

$$1 \;=\; \sum_{n=0}^{\infty} \left| d_n^{(\alpha)} \right|^2 \;=\; |c|^2 \sum_{n=0}^{\infty} \frac{\alpha^2}{n!} \;=\; |c|^2\, e^{\alpha^2} \,. \tag{4.144}$$

Normalization requires $c \,=\, \exp(-\alpha^2/2)$ and, hence, the quasi-classical states are

$$\phi^{(\alpha)}(y) \;=\; \exp\left(-\frac{\alpha^2}{2} \right) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}\, \phi_n(y) \,, \tag{4.145}$$

Using the generating function in the form (4.109), i.e.,

$$[\pi]^{-\frac{1}{4}}\, e^{-\frac{y^2}{2} + \sqrt{2}\,\alpha\, y - \alpha^2} \;=\; \exp\left(-\frac{\alpha^2}{2} \right) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}\, \phi_n(y) \,, \tag{4.146}$$

one can write

$$\phi^{(\alpha)}(y) \;=\; [\pi]^{-\frac{1}{4}} \exp\left(-\frac{1}{2}(y \;-\; \sqrt{2}\,\alpha)^2 \right) \tag{4.147}$$

which identifies this state as a ground state of the oscillator displaced by $\sqrt{2}\alpha$. This interpretation justifies the choice of $\alpha \in \mathbb{R}$ in (4.138). However, we will see below that $\alpha \in \mathbb{C}$ are also admissible and will provide a corresponding interpretation.
One can also write (4.145) using (4.84)

$$\phi^{(\alpha)}(y) \;=\; \exp\left(-\frac{\alpha^2}{2} \right) \sum_{n=0}^{\infty} \frac{(\alpha \hat{a}^+)^n}{n!}\, \phi_0(y) \;=\; e^{-\frac{\alpha\alpha^*}{2}}\, e^{\alpha \hat{a}^+}\, \phi_0(y) \,. \tag{4.148}$$

The identity (4.108) allows one to write

$$\phi^{(\alpha)}(y) \;=\; T^+(\alpha)\, \phi_0(y) \;=\; T^+(\alpha)\, [\pi]^{-\frac{1}{4}} e^{-y^2/2} \tag{4.149}$$

where

$$T^+(\alpha) \;=\; e^{\alpha \hat{a}^+ \,-\, \alpha^* \hat{a}^-} \tag{4.150}$$

for α presently real. Comparing (4.149) with (4.147) allows one to interprete $T(\alpha)$ as an operator which shifts the ground state by $\sqrt{2}\alpha$.

The shift operator as defined in (4.150) for complex α obeys the unitary property

$$T^{+}(\alpha)\, T(\alpha) \;=\; 1\!\!1 \tag{4.151}$$

which follows from a generalization of (4.95) to linear combinations $\alpha\hat{a}^{+} + \beta\hat{a}^{-}$ and reads then

$$\langle f(\alpha\hat{a}^{+} + \beta\hat{a}^{-})\, \phi|\psi\rangle \;=\; \langle\phi|f(\alpha^{*}\hat{a}^{-} + \beta^{*}\hat{a}^{+})\, \psi\rangle \;. \tag{4.152}$$

Since the operator function on the r.h.s. is the adjoint of the operator function on the l.h.s. one can write in case of $T^{+}(\alpha)$

$$\big(T^{+}(\alpha)\big)^{+} \;=\; T(\alpha) \;=\; e^{\alpha^{*}\hat{a}^{-} - \alpha\hat{a}^{+}} \;=\; e^{-(\alpha\hat{a}^{+} - \alpha^{*}\hat{a}^{-})} \;. \tag{4.153}$$

This is obviously the inverse of $T^{+}(\alpha)$ as defined in (4.150) and one can conclude that $T^{+}(\alpha)$ is a so-called unitary operator with the property

$$T^{+}(\alpha)\, T(\alpha) \;=\; T(\alpha)\, T^{+}(\alpha) \;=\; 1\!\!1 \tag{4.154}$$

or applying this to a scalar product like (4.152)

$$\langle T^{+}(\alpha)\, f\, |T^{+}(\alpha)\, g\rangle \;=\; \langle f\, |T(\alpha)\, T^{+}(\alpha)\, g\rangle \;=\; \langle f\, |g\rangle \;. \tag{4.155}$$

In particular, it holds

$$\langle T^{+}(\alpha)\, \phi_0\, |T^{+}(\alpha)\, \phi_0\rangle \;=\; \langle f\, |T(\alpha)\, T^{+}(\alpha)\, g\rangle \;=\; \langle\phi_0\, |\phi_0\rangle \;, \quad \alpha \in \mathbb{C} \tag{4.156}$$

, i.e., the shift operator leaves the ground state normalized.
Following the procedure adopted in determining the propagator of the harmonic oscillator [see (4.67, 4.68)] one can write the time-dependent solution with (4.145) as the initial state

$$\phi^{(\alpha)}(y,t_1) \;=\; \exp\left(-\frac{\alpha^2}{2}\right) \sum_{n=0}^{\infty} \exp\left(-i\,\omega\,(\,n \,+\, \frac{1}{2}\,)\,(\,t_1 \,-\, t_0\,)\right) \frac{\alpha^n}{\sqrt{n!}}\, \phi_n(y) \tag{4.157}$$

or

$$\phi^{(\alpha)}(y,t_1) \;=\; u^{\frac{1}{2}} \exp\left(-\frac{\alpha^2}{2}\right) \sum_{n=0}^{\infty} \frac{(\alpha\, u)^n}{\sqrt{n!}}\, \phi_n(y) \;, \quad u \;=\; e^{-i\omega(t_1 - t_0)} \;. \tag{4.158}$$

Comparision of this expression with (4.145) shows that α in (4.145) is replaced by a complex number αu which at times $t_1 = t_0 + 2\pi n/\omega$, $n = 0, 1, \ldots$ becomes real. Relationship (4.109) serves again to simplify this sum

$$\phi^{(\alpha)}(y,t_1) \;=\; \pi^{-\frac{1}{4}}\, u^{\frac{1}{2}} \exp\left(\underbrace{-\frac{y^2}{2} \,+\, \sqrt{2}\, y\, \alpha\, u \,-\, \frac{\alpha^2\,(u^2 \,+\, 1)}{2}}_{E}\right) \tag{4.159}$$

Noticing the identity which holds for $|u|^2 = 1$

$$u^2 \,+\, 1 \;=\; (\mathrm{Re}\, u)^2 \,-\, (\mathrm{Im}\, u)^2 \,+\, 1 \,+\, 2\, i\, \mathrm{Re}\, u\, \mathrm{Im}\, u \;=\; 2\, (\mathrm{Re}\, u)^2 \,+\, 2\, i\, \mathrm{Re}\, u\, \mathrm{Im}\, u \tag{4.160}$$

the exponent E can be written

$$\begin{aligned} E &= -\frac{y^2}{2} + \sqrt{2}\,\alpha y\,\mathrm{Re}u - \alpha^2\,(\mathrm{Re}\,u)^2 + i\,\sqrt{2}\,\alpha y\,\mathrm{Im}u - i\,\alpha^2\,\mathrm{Re}u\,\mathrm{Im}u \\ &= \underbrace{-\frac{1}{2}\left(y - \sqrt{2}\alpha\mathrm{Re}u\right)^2}_{\text{displaced Gaussian}} + \underbrace{i\,\sqrt{2}y\alpha\,\mathrm{Im}u}_{\text{momentum}} - \underbrace{i\,\alpha^2\mathrm{Re}u\,\mathrm{Im}u}_{\text{phase term}} \end{aligned} \tag{4.161}$$

and is found to describe a displaced Gaussian, a momentum factor and a phase factor. It is of interest to note that the displacement y_0 of the Gaussian as well as the momentum k_0 and phase ϕ_0 associated with (4.161) are time-dependent

$$\begin{aligned} y_0(t_1) &= \sqrt{2}\,\alpha\cos\omega(t_1 - t_0) \\ k_0(t_1) &= -\sqrt{2}\,\alpha\sin\omega(t_1 - t_0) \\ \phi_0(t_1) &= -\frac{1}{2}\alpha^2\sin 2\omega(t_1 - t_0) \end{aligned} \tag{4.162}$$

with a periodic change. The oscillations of the mean position $y_0(t_1)$ and of the mean momentum $k_0(t_1)$ are out of phase by $\frac{\pi}{2}$ just as in the case of the classical oscillator. Our interpretation explains also the meaning of a complex α in (4.138): a real α corresponds to a displacement such that initially the oscillator is at rest, a complex α corresponds to a displaced oscillator which has initially a non-vanishing velocity; obviously, this characterization corresponds closely to that of the possible initial states of a classical oscillator. The time-dependent wave function (4.159) is then

$$\phi^{(\alpha)}(y, t_1) = \pi^{-\frac{1}{4}}\exp\left(-\frac{1}{2}[y - y_0(t_1)]^2 + iy\,k_0(t_1) - i\phi_0(t_1) - \frac{i}{2}\omega(t_1 - t_0)\right) . \tag{4.163}$$

This solution corresponds to the initial state given by (4.138, 4.147).
In Fig. 4.1 (top row) we present the probability distribution of the Glauber state for various instances in time. The diagram illustrates that the wave function retains its Gaussian shape with constant width for all times, moving solely its center of mass in an oscillator fashion around the minimum of the harmonic potential. This shows clearly that the Glauber state is a close analogue to the classical oscillator, except that it is not pointlike.
One can express (4.163) also through the shift operator. Comparing (4.148, 4.150) with (4.158) allows one to write

$$\phi^{(\alpha)}(y, t_1) = u^{\frac{1}{2}}\,T^+(\alpha\,u)\,\phi_0(y) \tag{4.164}$$

where according to (4.150)

$$T^+(\alpha\,u) = \exp\left(\alpha\,(\,e^{-i\omega(t_1-t_0)}\hat{a}^+ - e^{i\omega(t_1-t_0)}\hat{a}^-\,)\right) . \tag{4.165}$$

This provides a very compact description of the quasi-classical state.

Arbitrary Gaussian Wave Packet Moving in a Harmonic Potential

We want to demonstrate now that any initial state described by a Gaussian shows a time dependence very similar to that of the Glauber states in that such state remains Gaussian, being displaced around the center with period $2\pi/\omega$ and, in general, experiences a change of its width (relative to

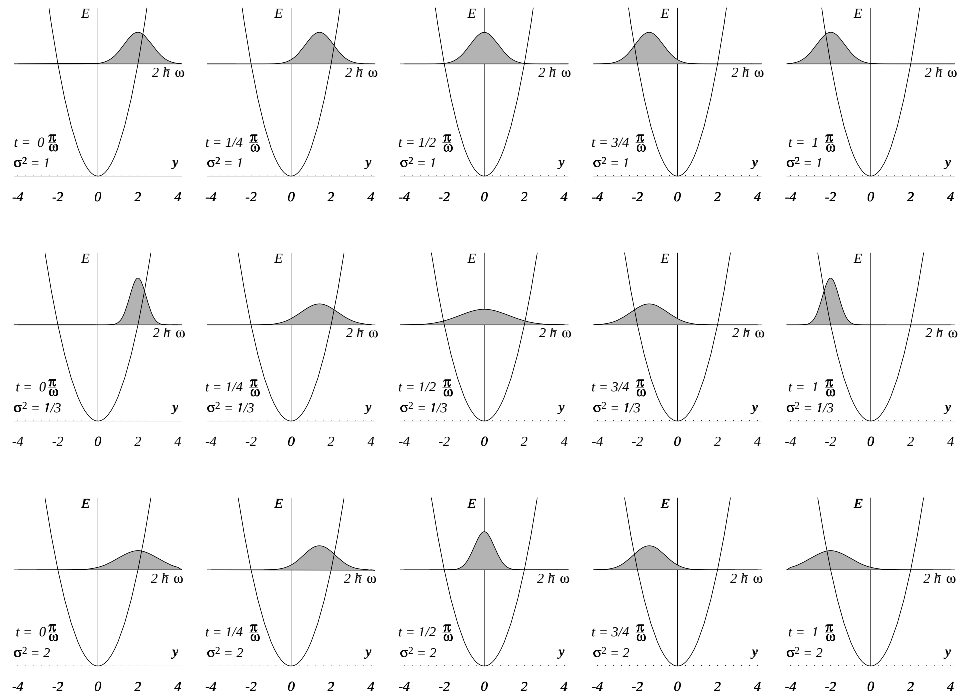

Figure 4.1: Time dependence of Gaussian wave packets in harmonic oscillator potential.

that of the Glauber states) with a period π/ω. This behaviour is illustrated in Fig. 4.1 (second and third row). One can recognize the cyclic displacement of the wave packet and the cyclic change of its width with a period of twice that of the oscillator.
To describe such state which satisfies the initial condition

$$\phi(a_o, \sigma|y, t_0) \;=\; [\pi\sigma^2]^{-\frac{1}{4}} \exp\left(- \frac{(y-a_o)^2}{2\sigma^2} \right) \tag{4.166}$$

we expand

$$\phi(a_o, \sigma|y, t_0) \;=\; \sum_{n=0}^{\infty} d_n \, \phi_n(y) \; . \tag{4.167}$$

One can derive for the expansion coefficients

$$\begin{aligned} d_n \;&=\; \int_{-\infty}^{+\infty} dy \, \phi_n(y) \, \phi(a_o, \sigma|y, t_0) \\ &=\; \frac{1}{\sqrt{2^n n! \pi\sigma}} \int_{-\infty}^{+\infty} dy \exp\left(- \frac{(y-a_o)^2}{2\sigma^2} - \frac{y^2}{2} \right) H_n(y) \\ &=\; \frac{e^{-a_o q/2}}{\sqrt{2^n n! \pi\sigma}} \int_{-\infty}^{+\infty} dy \, e^{-p(y-q)^2} \, H_n(y) \end{aligned} \tag{4.168}$$

where

$$p \;=\; \frac{1 + \sigma^2}{2\sigma^2} \; , \quad q \;=\; \frac{a}{1 + \sigma^2} \; . \tag{4.169}$$

The solution of integral (4.168) is[4]

$$d_n \; = \; \frac{e^{-aq/2}}{\sqrt{2^n n! \sigma}} \, \frac{(p-1)^{n/2}}{p^{(n+1)/2}} \, H_n(q\sqrt{\frac{p}{p-1}}) \; . \tag{4.170}$$

In order to check the resulting coefficients we consider the limit $\sigma \to 1$. In this limit the coefficients d_n should become identical to the coefficients (4.143) of the Glauber state, i.e., a ground state shifted by $a_o = \sqrt{2}\alpha$. For this purpose we note that according to the explicit expresssion (**??**) for $H_n(y)$ the leading power of the Hermite polynomial is

$$H_n(y) \; = \; (2\,y)^n \; + \; \begin{cases} O(y^{n-2}) & \text{for } n \geq 2 \\ 1 & \text{for } n = 0, 1 \end{cases} \tag{4.171}$$

and, therefore,

$$\lim_{\sigma\to 1} \left(\frac{p-1}{p} \right)^{\frac{n}{2}} H_n(q\sqrt{\frac{p}{p-1}}) \; = \; \lim_{\sigma\to 1} (2q)^n \; = \; a^n \; . \tag{4.172}$$

One can then conclude

$$\lim_{\sigma\to 1} d_n \; = \; \frac{e^{-a^2/4}}{\sqrt{2^n n!}} \, a^n \tag{4.173}$$

which is, in fact, in agreement with the expected result (4.143).
Following the strategy adopted previously one can express the time-dependent state corresponding to the initial condition (4.166) in analogy to (4.157) as follows

$$\phi(a_o, \sigma | y, t_1) \; = \tag{4.174}$$
$$\frac{e^{-aq/2}}{\sqrt{p\sigma}} \sum_{n=0}^{\infty} e^{-i\omega(n+\frac{1}{2})(t_1 - t_o)} \frac{1}{\sqrt{2^n n!}} \left(\frac{p-1}{p} \right)^{n/2} H_n(q\sqrt{\tfrac{p}{p-1}}) \, \phi_n(y) \; .$$

This expansion can be written

$$\phi(a_o, \sigma | y, t_1) \; = \; \frac{1}{\sqrt{p\sigma\sqrt{\pi}}} \exp\Big(-\tfrac{1}{2} \tfrac{a^2}{1+\sigma^2} \; + \; \tfrac{1}{2} y_o^2 \; - \; \tfrac{1}{2}\omega(t_1 - t_o) \Big) \; \times$$
$$\times \; \sum_{n=0}^{\infty} \frac{t^n}{2^n n!} \, H_n(y_o) \, e^{-y_o^2/2} \, H_n(y) \, e^{-y^2/2} \tag{4.175}$$

where

$$y_o \; = \; q\sqrt{\frac{p}{p-1}} \; = \; \frac{a}{\sqrt{1 \, - \, \sigma^4}} \; , \quad t \; = \; \sqrt{\frac{1 \, - \, \sigma^2}{1 \, + \, \sigma^2}} \, e^{-i\omega(t_1 - t_o)} \tag{4.176}$$

The generating function (4.76) permits us to write this

$$\phi(a_o, \sigma | y, t_1) \; = \; \frac{1}{\sqrt{p\sigma\sqrt{\pi}}} \, \frac{1}{\sqrt{1-t^2}} \; \times \tag{4.177}$$
$$\times \, \exp\Big[- \tfrac{1}{2}(y^2 \, + \, y_o^2) \, \tfrac{1+t^2}{1-t^2} \; - \; \tfrac{1}{2} \tfrac{a^2}{1+\sigma^2} \; + \; \tfrac{1}{2}\, y_o^2 \; + \; 2\, y\, y_o \, \tfrac{t}{1-t^2} \; - \; \tfrac{1}{2}\omega(t_1 - t_o) \Big] \; .$$

[4]This integral can be found in *Integrals and Series, vol. 2* by A.P. Prudnikov, Yu. Brychkov, and O.I. Marichev (Wiley, New York, 1990); this 3 volume integral table is likely the most complete today, a worthy successor of the famous Gradshteyn, unfortunately very expensive, namely $750