Solutions to Problem Set 2
Physics 480 / Fall 1999
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(a)

$$L(x, \dot{x}, \tau) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2 + xF(\tau) \tag{1}$$

The classical equation of motion is given by the Euler-Lagrange equation

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \quad \Rightarrow \quad \ddot{x} = -\omega^2 x + \frac{F(\tau)}{m} \ . \tag{2}$$

(b) The function $\xi = \dot{x} + \mathrm{i}\omega x$ obeys the following differential equation

$$\dot{\xi} = \ddot{x} + \mathrm{i}\omega\dot{x} \overset{(2)}{=} \mathrm{i}\omega\xi + \frac{F(\tau)}{m} \ , \tag{3}$$

with the general solution

$$\xi(\tau) = \left[\xi_0 + \frac{1}{m}\int_{t_0}^{\tau} ds F(s) \mathrm{e}^{-\mathrm{i}\omega s}\right] \mathrm{e}^{\mathrm{i}\omega t} \ . \tag{4}$$

Now

$$x(\tau) = \frac{1}{\omega}\mathrm{Im}\{\xi\} \overset{(4)}{=} a\sin\omega\tau + b\cos\omega\tau + \frac{1}{m\omega}\int_{t_0}^{\tau} ds F(s)\sin\omega(\tau - s) \ , \tag{5}$$

where the constants a and b are related to ξ_0 and can be determined by using the boundary conditions $x(\tau = t_0) = x_0$ and $x(\tau = t) = x$. One obtains

$$\begin{aligned} a &= \frac{x\cos\omega t_0 - x_0\cos\omega t}{\sin\omega(t - t_0)} - \frac{\cos\omega t_0}{m\omega\sin\omega(t - t_0)}\int_{t_0}^{t} ds F(s)\sin\omega(t - s) \ , \\ b &= \frac{x_0\sin\omega t - x\sin\omega t_0}{\sin\omega(t - t_0)} - \frac{\sin\omega(\tau - t_0)}{m\omega\sin\omega(t - t_0)}\int_{t_0}^{t} ds F(s)\sin\omega(t - s) \ . \end{aligned} \tag{6}$$

Thus, the corresponding classical path is given by

$$\begin{aligned} x_{cl}(\tau) = {} & \frac{x_0\sin\omega(t - \tau) + x\sin\omega(\tau - t_0)}{\sin\omega(t - t_0)} - \frac{\sin\omega(\tau - t_0)}{m\omega\sin\omega(t - t_0)} \times \\ & \times \int_{t_0}^{t} ds F(s)\sin\omega(t - s) + \frac{1}{m\omega}\int_{t_0}^{\tau} ds F(s)\sin\omega(\tau - s) \ . \end{aligned} \tag{7}$$

The first term on the RHS in (7) corresponds to the classical path of the harmonic oscillator in the absence of the driven force F, i.e.,

$$x_{cl}^{F=0}(\tau) = \frac{x_0\sin\omega(t - \tau) + x\sin\omega(\tau - t_0)}{\sin\omega(t - t_0)} \ . \tag{8}$$

The last two terms on the RHS in (7) can be transformed as follows

$$-\frac{\sin\omega(\tau-t_0)}{m\omega\sin\omega(t-t_0)}\int_{t_0}^{t} dsF(s)\sin\omega(t-s)-\frac{1}{m\omega\sin\omega(t-t_0)}\int_{t_0}^{\tau} dsF(s)\times$$

$$\times[\sin\omega(\tau-t_0)\sin\omega(t-s)-\sin\omega(t-t_0)\sin\omega(\tau-s)]=$$

$$=-\frac{1}{m\omega\sin\omega(t-t_0)}\left[\int_{t_0}^{\tau} dsF(s)\sin\omega(t-\tau)\sin\omega(s-t_0)+\right.$$

$$\left.+\int_{\tau}^{t} dsF(s)\sin\omega(t-s)\sin\omega(\tau-t_0)\right] , \tag{9}$$

where we have used the identity

$$\sin\omega(\tau-t_0)\sin\omega(t-s)-\sin\omega(t-t_0)\sin\omega(\tau-s)=\sin\omega(t-\tau)\sin\omega(s-t_0) .$$

Finally

$$x_{cl}(\tau)=x_{cl}^{F=0}(\tau)-\frac{1}{m\omega}\int_{t_0}^{t} dsg(\tau,s)F(s) , \tag{10}$$

where

$$g(\tau,s)=\begin{cases}\frac{\sin\omega(t-\tau)\sin\omega(s-t_0)}{\sin\omega(t-t_0)} & \text{for} \quad s\le\tau\\ \frac{\sin\omega(t-s)\sin\omega(\tau-t_0)}{\sin\omega(t-t_0)} & \text{for} \quad s>\tau .\end{cases} \tag{11}$$

(c) Our Lagrangian (1) coincides with the one given by Eq.(128) in the lecture notes provided that we take $c(t)=m\omega^2$ and $e(t)=-F(t)$. Therefore, we can use directly the result obtained in the lecture notes for the propagator $\phi(x,t|x_0,t_0)$, i.e.,

$$\phi(x,t|x_0,t_0)=\left[\frac{m}{2\pi i\hbar f(t,t_0)}\right]^2\exp\left\{\frac{i}{\hbar}S[x_{cl}(\tau)]\right\} , \tag{12}$$

where $f(t,t_0)$ is the solution of the following problem (see Eqs. (148)-(149) in the lecture notes)

$$\begin{cases}\frac{d^2f}{dt^2}=-\frac{c(t)}{m}f=-\omega^2 f ,\\ f(t_0,t_0)=0 \quad \text{and} \quad f'(t_0,t_0)=1 .\end{cases} \tag{13}$$

Hence

$$f(t_0,t)=\frac{1}{\omega}\sin\omega(t-t_0) . \tag{14}$$

Pluging (14) into Eq.(12) one obtains the desired result. Note that the effect of the external force F shows up only in $S[x_{cl}(\tau)]$ but not in the function $f(t_0,t)$ which has the same expression as that corresponding to the unperturbed harmonic oscillator.

(d)

$$S[x_{cl}] = \int_{t_0}^{t}\Big[\frac{m\dot{x}_{cl}^2}{2} - \frac{m\omega^2 x_{cl}^2}{2} + x_{cl}F\Big]\,d\tau =$$

$$= \frac{m}{2}\int_{t_0}^{t} d(x_{cl}\dot{x}_{cl}) - \frac{m}{2}\int_{t_0}^{t}(x_{cl}\ddot{x}_{cl} + \omega^2 x_{cl}^2 - \frac{2}{m}x_{cl}F)d\tau \overset{(2)}{=}$$

$$= \frac{m}{2}[x\dot{x}_{cl}(t) - x_0\dot{x}_{cl}(t_0)] + \frac{1}{2}\int_{t_0}^{t} x_{cl}(\tau)F(\tau)d\tau\ . \qquad (15)$$

Above, firstly we partially integrated $\frac{m\dot{x}_{cl}^2}{2}$, and secondly, we expressed $\ddot{x}_{cl}$ in terms of the equation of motion (2). On the other hand, Eqs.(11) and (12) yield

$$\dot{x}_{cl}(t_0) = \omega\frac{x - x_0\cos\omega(t-t_0)}{\sin\omega(t-t_0)} - \frac{1}{m\sin\omega(t-t_0)}\int_{t_0}^{t} dsF(s)\sin\omega(t-s)\ ,$$

$$\dot{x}_{cl}(t) = \omega\frac{x\cos\omega(t-t_0) - x_0}{\sin\omega(t-t_0)} + \frac{1}{m\sin\omega(t-t_0)}\int_{t_0}^{t} dsF(s)\sin\omega(s-t_0)\ . \qquad (16)$$

Inserting now Eqs.(16) and (10) in (15), after some simple algebra, one obtains the desired result

$$S[x_{cl}] = \frac{m\omega}{2\sin\omega(t-t_0)}[(x^2+x_0^2)\cos\omega(t-t_0) - 2xx_0] +$$
$$+x\int_{t_0}^{t} dsF(s)\frac{sin\omega(s-t_0)}{\sin\omega(t-t_0)} + x_0\int_{t_0}^{t} dsF(s)\frac{\sin\omega(t-s)}{\sin\omega(t-t_0)} - \qquad (17)$$
$$-\frac{1}{2m\omega}\int_{t_0}^{t} d\tau\int_{t_0}^{t} dsF(\tau)g(\tau,s)F(s)\ .$$

(e) Inserting $F(\tau) = F_0\theta(\tau)$ ($\theta(\tau)$ is the step function, i.e., $\theta(\tau) = 1$ for $\tau > 0$ and $\theta(\tau) = 0$ for $\tau < 0$) into Eq.(17) and carrying out the integrals one finds that

$$S[x_{cl}] = \frac{m\omega}{2\sin\omega t}[(x^2+x_0^2)\cos\omega t - 2xx_0] + \frac{F_0(x+x_0)}{\omega\sin\omega t}(1-\cos\omega t) -$$
$$-\frac{F_0^2}{m\omega^3\sin\omega t}(1-\cos\omega t) + \frac{F_0^2 t}{2m\omega^2}\ .$$

This, with $F_0 = m\omega^2 a$, yields

$$S[x_{cl}] = \frac{m\omega}{2\sin\omega t}[(x^2+x_0^2)\cos\omega t - 2xx_0 + 2a(x+x_0)(1-\cos\omega t) -$$
$$-2a^2(1-\cos\omega t)] + \frac{m\omega^2 a^2 t}{2} = \qquad (18)$$
$$= \frac{m\omega}{2\sin\omega t}\{[(x-a)^2 + (x_0-a)^2]\cos\omega t - 2(x-a)(x_0-a)\} + \frac{m\omega^2 a^2 t}{2}$$

Q.E.D.

The harmonic oscillator driven by the constant external force F_0 is equivalent with a free but displaced harmonic oscillator. The displacement of the oscillator is a, i.e., the classical equilibrium position is $x = a$ and not $x = 0$.

(f)

$$\Psi(x,t) = \int_{-\infty}^{\infty} dx_0 \phi(x,t|x_0,0)\Psi_0(x_0,0) =$$
$$= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \exp\left(\mathrm{i}\frac{m\omega^2 a^2}{2\hbar}t\right)\left(\frac{m\omega}{2\pi\mathrm{i}\hbar\sin\omega t}\right)^{\frac{1}{2}} \exp\left[\frac{\mathrm{i}m\omega}{2\hbar\sin\omega t}(x-a)^2\cos\omega t\right]\times$$
$$\times\int_{-\infty}^{\infty} dx_0 \exp\left[\frac{\mathrm{i}m\omega}{2\hbar\sin\omega t}\{(x-a)^2\cos\omega t - 2(x-a)(x_0-a)\} - \frac{m\omega}{2\hbar}x_0^2\right] . \tag{19}$$

The integral in (19) is Gaussian and can be easily evaluated by making use of the formula

$$\int_{-\infty}^{\infty} dx_0 e^{-Ax_0^2+Bx_0} = \sqrt{\frac{\pi}{A}}e^{\frac{B^2}{4A}} \qquad (\mathrm{Re}A \geq 0) . \tag{20}$$

The result of the integral is

$$\left(\frac{2\pi\mathrm{i}\hbar\sin\omega t}{m\omega}\right)^{\frac{1}{2}} e^{-\frac{\mathrm{i}\omega t}{2}} \exp\left[-\frac{\mathrm{i}m\omega}{2\hbar\sin\omega t}(x-a)^2\cos\omega t\right]\times$$
$$\times\exp\left[-\frac{m\omega}{2\hbar}(x-a)^2 - \frac{m\omega a}{\hbar}e^{-\mathrm{i}\omega t}(x-a) - \frac{m\omega a^2}{2\hbar}\cos\omega t e^{-\mathrm{i}\omega t}\right] . \tag{21}$$

Inserting (21) into (19) and keeping in mind that

$$-\cos\omega t e^{-\mathrm{i}\omega t} = -1 + \mathrm{i}\sin\omega t e^{-\mathrm{i}\omega t} ,$$

one obtains the desired result, namely

$$\Psi(x,t) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \exp\left[-\frac{\mathrm{i}\omega t}{2}\left(1 - \frac{m\omega a^2}{\hbar}\right)\right] \exp\left\{-\frac{m\omega}{2\hbar}(x-a)^2 - \right.$$
$$\left. -\frac{m\omega a}{\hbar}e^{-\mathrm{i}\omega t}(x-a) + \frac{\mathrm{i}\omega a^2}{2\hbar}\sin\omega t e^{-\mathrm{i}\omega t}\right\} \exp\left(-\frac{m\omega a^2}{2\hbar}\right) . \tag{22}$$

(g) In (22) we can switch to the new units by setting $\omega = 2\pi$ and $\hbar = m\omega$; the result is

$$\Psi(x,t) = \pi^{\frac{1}{4}} \exp[-\mathrm{i}\pi t(1-a^2)] \exp\{-\frac{1}{2}(x-a)^2 - a(x-a)e^{-\mathrm{i}2\pi t} +$$

$$+\frac{\mathrm{i}a^2}{2}\sin 2\pi t e^{-\mathrm{i}2\pi t}\}\exp\left(-\frac{a^2}{2}\right) . \tag{23}$$

Thus

$$P(x,t) \equiv |\Psi(x,t)|^2 = \Psi^*(x,t)\Psi(x,t) = \pi^{\frac{1}{2}}\exp[-(x-a)^2 - a^2 +$$

$$+a^2\sin^2 2\pi t - 2a(x-a)\cos 2\pi t] = \frac{1}{\sqrt{\pi}}\exp\left[-(x-a+a\cos 2\pi t)^2\right] \tag{24}$$

Q.E.D.

(h) By following the procedure described at part (b), we obtain the following classical law of motion for a harmonic oscillator driven by the force $F(t) = F_0\theta(t)$

$$x_{cl}(t) = A\sin\omega t + B\cos\omega t + \frac{F_0}{m\omega^2}(1-\cos\omega t) . \tag{25}$$

The integration constants A and B can be determined from the initial conditions $x_{cl}(0) = 0$ and $\dot{x}_{cl}(0) = 0$. The result is $A = B = 0$. Thus

$$x_{cl}(t) = \frac{F_0}{m\omega^2}(1-\cos\omega t) = a(1-\cos\omega t) , \tag{26}$$

where $a \equiv F_0/m\omega^2$. By comparing (24) and (26) one can infer that

$$P(x,t) = \frac{1}{\sqrt{\pi}}\exp\left\{-\left[x - x_{cl}(t)\right]^2\right\} . \tag{27}$$

This last equation tells us that the probability of finding the oscillator in the spatial interval $(x, x+dx)$ at the instant of time t is maximum for $x = x_{cl}(t)$. In other words, the maximum of the probability density (24)-(27) evolves in time according to the classical law of motion (26).

(i) The potential energy $V(x)$ can be rewritten as

$$V(x) = \frac{1}{2}m\omega^2x^2 - F_0x = \frac{1}{2}m\omega^2(x-a)^2 - \frac{F_0a}{2} , \tag{28}$$

where $a = F_0/m\omega^2$. By making the change of variable $y = x - a$, the Hamiltonian becomes

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dy^2} + \frac{m\omega^2}{2}y^2 - \frac{F_0a}{2} . \tag{29}$$

The lowest eigenvalue of $\hat{H}$ (i.e., the ground state energy) is given by

$$E_0^F = \frac{\hbar\omega}{2} - \frac{F_0a}{2} = \frac{\hbar\omega}{2}\left(1 - \frac{m\omega}{\hbar}a^2\right) , \tag{30}$$

or, by employing the new length and time units ($L = \sqrt{\hbar/m\omega}$ and $T = 2\pi/\omega$),

$$E_0^F = \hbar\pi\left(1 - a^2\right) \ . \tag{31}$$

The corresponding stationary wave function reads

$$\Psi_0^F(x,t) = \pi^{-\frac{1}{4}} \exp\left(-\frac{1}{2}y^2\right) e^{-\mathrm{i}\pi t\left(1-a^2\right)} \ , \tag{32}$$

where $y = x - a$. This is evidently the ground state wave function of a displaced harmonic oscillator. $x = a$ gives the classical equilibrium position of the oscillator.

(j) One has

$$P_0 = \left|\int_{-\infty}^{\infty} dx \left[\Psi_0^F(x,t)\right]^* \Psi(x,t)\right|^2 = \frac{1}{\pi}\left|\int_{-\infty}^{\infty} dx \exp\left[-(x-a)^2 - ae^{-2\pi\mathrm{i}t}(x-a)\right]\right|^2 \times$$

$$\times \exp(-a^2 + a^2\sin^2 2\pi t) = \frac{1}{\pi}\exp(-a^2\cos^2 2\pi t)\pi\exp\left(\frac{a^2}{2}\cos 4\pi t\right) =$$

$$= \exp(-a^2\cos^2 2\pi t)\cdot\exp\left(a^2\cos^2 2\pi t - \frac{a^2}{2}\right) = \exp\left(-\frac{a^2}{2}\right) \ . \tag{33}$$

(k) Since the Hamiltonian (29) is that corresponding to a displaced harmonic oscillator, one can write down imediately the corresponding eigenfunctions

$$\Psi_n^F(x,t) = \pi^{-\frac{1}{4}}\frac{1}{2^{2/n}\sqrt{n!}}e^{-\frac{y^2}{2}}H_n(y) \ , \tag{34}$$

and energy eigenvalues

$$E_n^F = \frac{\hbar\omega}{2}(2n+1) - \frac{F_0 a}{2} \ , \qquad n = 0, 1, 2, \ldots \tag{35}$$

(l) By definition

$$P_n = \left|\int_{-\infty}^{\infty} dx \left[\Psi_n^F(x,t)\right]^* \Psi(x,t)\right|^2 = \frac{1}{\pi}\frac{1}{2^n n!}\left|\int_{-\infty}^{\infty} dx H_n(x-a)\times\right.$$

$$\left.\times\exp\left[-(x-a)^2 - ae^{-2\pi\mathrm{i}t}(x-a)\right]\right|^2 \exp(-a^2 + a^2\sin^2 2\pi t) \ . \tag{36}$$

The integral in Eq.(36) can be evaluated as follows. First, change the integration variable: $y = x - a$. Then, by setting $z = -\frac{a}{2}e^{-2\pi t}$, one has

$$\int_{-\infty}^{\infty} dy H_n(y) e^{-y^2+2zy} = \left[\int_{-\infty}^{\infty} dy e^{-y^2} H_n(y) e^{2zy-z^2}\right] e^{z^2} =$$

$$= e^{z^2} \int_{-\infty}^{\infty} dy e^{-y^2} H_n(y) \sum_{m=0}^{\infty} \frac{H_m(y)}{m!} z^m = e^{z^2} \sum_{m=0}^{\infty} \frac{z^m}{m!} \times \tag{37}$$

$$\times \int_{-\infty}^{\infty} dy e^{-y^2} H_n(y) H_m(y) = e^{z^2} \sum_{m=0}^{\infty} \frac{z^m}{m!} 2^n n! \sqrt{\pi} \delta_{nm} =$$

$$= 2^n \sqrt{\pi} z^n e^{z^2} = \sqrt{\pi} (-a)^n e^{-2\pi i n t} \exp \left[\left(\frac{a}{2} \right)^2 e^{-4\pi \mathrm{i} t} \right] .$$

Thus

$$P_n = \frac{\left(a^2/2\right)^n}{n!} \exp\left(\frac{a^2}{2} \cos 4\pi t \right) \exp(-a^2 \cos^2 2\pi t) =$$

$$= \frac{\left(a^2/2\right)^n}{n!} \exp\left(-\frac{a^2}{2} \right) . \tag{38}$$

Q.E.D.

Since $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$, it follows that

$$\sum_{n=0}^{\infty} P_n = e^{-\frac{a^2}{2}} \sum_{n=0}^{\infty} \frac{(a^2/2)^n}{n!} = e^{-\frac{a^2}{2}} e^{\frac{a^2}{2}} = 1 . \tag{39}$$

(m) The required plots are given in Figure.1 and Figure.2, respectively. From these plots one can infer

(i) For a given a, if $a \leq 1$, P_n is a monotonically decreasing function of n whilst, if $a > 1$, P_n first increases to a maximum value and then goes rapidly to zero.

(ii) For $n = 0$, $P_0(a)$ decreases exponentially from $P_n(0) = 1$ to zero as a goes to infinity whilst, for $n \geq 1$, $P_n(a)$ first increases from $P_n(0) = 0$ to its maximum value (which ca be determined by solving the equation $dP_n(a)/da = 0$) and then decays exponentially to zero.

(n) Those values of a for which the chance to find the oscillator, at $t > 0$, in its second excited state (i.e., that corresponding to $n = 2$) will be larger than the probability to find the oscillator in any other state can be obtained by imposing the condition $P_1(a) < P_2(a) < P_3(a)$. After some algebra one finds $2 < a < \sqrt{6}$, and consequently $F_0 \in (2m\omega^2, \sqrt{6}m\omega^2)$.

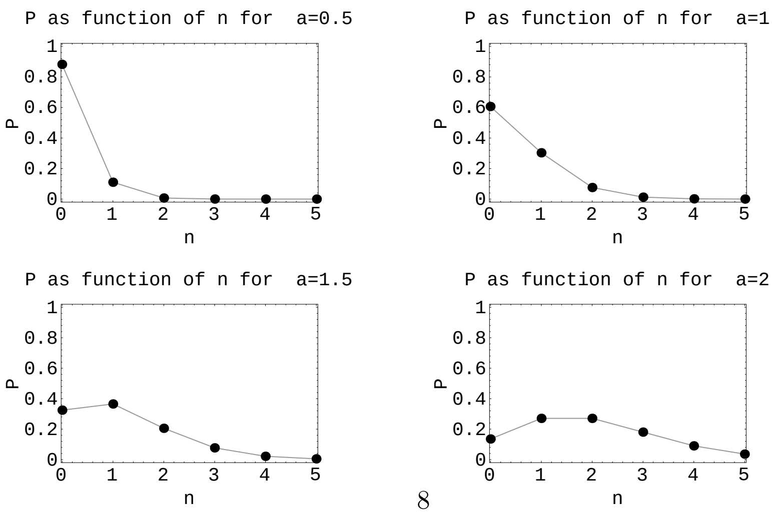

Figure 1: P_n as function of n for $a \in \{0.5, 1.0, 1.5, 2.0\}$ and $n \leq 5$.

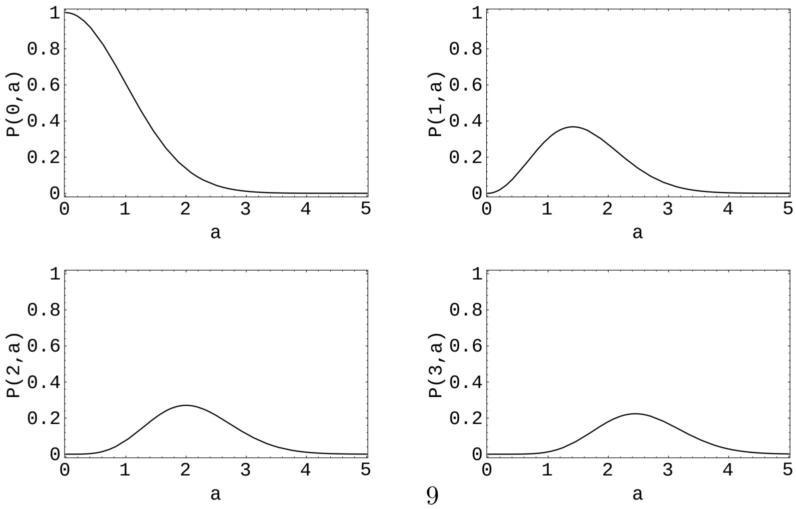

Figure 2: P_0, P_1, P_2 and P_3 as a function of $a \in [0, 5]$.