Wave packets (February 17)

We want to use quantum mechanics to describe a localized particle. How do we describe the particle's wave function? We have already found the stationary states for free particles, so any wave function must be combination of these states:

$$\psi(x) \; = \; \int_{-\infty}^{\infty} dk \; a(k;k_o) \; e^{ik(x-x_o)} \tag{1}$$

You can think of the function $a(k;k_o)$ as a weight factor; the entire wave function consists of the superposition of a bunch of free particle wave function with k-vector k. If we take the weight factor to have a Gaussian form,

$$a(k;k_o) \; = \; N \; exp\left(-\frac{(k-k_o)^2}{2(\Delta k)^2}\right), \tag{2}$$

where N is some normalization constant that does not depend on k, then our wave function will have momenta centered around $\hbar k_o$.

When we do the integral, we get

$$\psi(x) \; \propto \; exp\left((x-x_o)^2\frac{\Delta k^2}{2} + ik_o(x-x_o)\right). \tag{3}$$

So the probability distribution has $\Delta x \sim \frac{1}{\Delta k}$. This is a manifestation of the Heisenberg Uncertainty Principle:

$$\Delta x \Delta p \geq \hbar. \tag{4}$$

The motion of wave packets (February 19) (See also page 59 ff of Goswami)

The time evolution of our wave packet is governed by the time-dependent Schroedinger equation:

$$i\hbar\partial_t\psi(x,t) \; = \; -\frac{\hbar^2}{2m}\partial_x^2\psi(x,t). \tag{5}$$

When we studied the particle in a box, we found that when the wave function was a superposition of several different stationary states, interference effects led to a nontrivial time depedendence for the probability distribution. The same thing will happen here, since the wave packet is the (Gaussian- weighted) sum of many plane-wave stationary states.

In fact, because the exponential has an imaginary part that depends on x, we can see already that there will be a probability current associated with the wave packet.

Dirac delta functions

What is the Fourier transform of the number 1? Let's find out. Let $F(k) = 1$, so that it's Fourier transform, $f(x)$, is given by

$$f(x) = \int dk\ F(k)\ e^{ikx} \tag{6}$$

$$= \int dk\ e^{ikx}. \tag{7}$$

Then, for the inverse Fourier transform, we must have

$$1 = \frac{1}{2\pi} \int dx\ f(x)\ e^{-ikx}. \tag{8}$$

The function $f(x)$ must have the property of being 0 when $x \neq 0$, but infinite in some way when $x = 0$, so that the entire integral evaluates to 1. We call such a function a *delta function*, $\delta(x)$, and they are extremely useful in physics. They are really only defined by what they do inside integrals: if an integral has a delta function, the integral goes away and you evaluate the integrand at the point(s) where the argument in the delta function is zero:

$$\int f(x)\ \delta(x)\ dx = f(0); \tag{9}$$

$$\int f(x)\ \delta(x - x_o)\ dx = f(x_o). \tag{10}$$

We have also learned that one example of a delta function is

$$\delta(x) = \frac{1}{2\pi} \int dk\ e^{ikx}. \tag{11}$$

This definition allows us to normalize our wave function.

Time dependent solution

In Goswami, p. 41, the time dependent wave function for the Gaussian wave packet is shown. We can write the probability distribution as a function of time in the form

$$P(x,t) = N\ exp\left[-\frac{(x - \frac{p_o}{m}t)^2}{\delta^2(1 + \frac{\hbar^2 t^2}{m^2 \delta^4})} \right] \tag{12}$$

Here $\delta = 1/\Delta k$ and $p_o = \hbar k_o$. From this we can see that the probability distribution at time t is centered at p_o/m, corresponding to a classical velocity. The width, which proportional to the square root of the denominator in the exponent, increases linearly in time for large times. The mass of the particle, compared to $\hbar$, and the initial size of the wave packet determine how quickly the probability distribution will spread out.