Consider a general hamiltonian, assumed hermitian, in a basis of two states. The hamiltonian has the form

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \tag{1}$$

in some basis $\{|1\rangle, |2\rangle\}$. If these basis states were eigenstates of the hamiltonian, then the off-diagonal elements of H would equal zero, and the eigenvalues would be the diagonal elements. We assume that these states in fact are not the eigenstates of H, and now try to find what eigenstates actually are.

We can write any state in this system in matrix format as

$$|\psi\rangle \rightarrow \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{2}$$

where $\alpha = \langle 1|\psi\rangle$ is the amplitude to be found in state $|1\rangle$ and $\beta = \langle 2|\psi\rangle$ is the amplitude to be found in $|2\rangle$. We want to find eigenstates of the hamiltonian, with amplitudes α_1, β_1 and α_2, β_2, and corresponding energy eigenvalues E_1, E_2. If we call the eigenstates $\left|\tilde{1}\right\rangle$ and $\left|\tilde{2}\right\rangle$, then the equation we need to solve is

$$\hat{H}\left|\tilde{j}\right\rangle = E_j \left|\tilde{j}\right\rangle \ , \tag{3}$$

which in matrix notation has the form

$$\begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \begin{pmatrix} \alpha_j \\ \beta_j \end{pmatrix} = E_j \begin{pmatrix} \alpha_j \\ \beta_j \end{pmatrix} \tag{4}$$

To solve this matrix equation, we subtract the right hand side from the left, and set the determinant of the resulting matrix equal to zero:

$$\det \begin{pmatrix} H_{11} - E & H_{12} \\ H_{21} & H_{22} - E \end{pmatrix} = 0 \ . \tag{5}$$

This gives us a quadratic equation for the energies E:

$$\begin{aligned} (H_{11} - E)(H_{22} - E) - |H_{12}|^2 &= 0 \ ; \\ E^2 - E(H_{11} + H_{22}) - |H_{12}|^2 + H_{11}H_{22} &= 0 \ . \end{aligned} \tag{6}$$

Using the quadratic formula, we see that the two roots of this equation are

$$\begin{aligned} E &= \frac{H_{11} + H_{22}}{2} \pm \sqrt{\left(\frac{H_{11} + H_{22}}{2}\right)^2 + |H_{12}|^2 - H_{11}H_{22}} \\ &= \frac{H_{11} + H_{22}}{2} \pm \sqrt{\left(\frac{H_{11} - H_{22}}{2}\right)^2 + |H_{12}|^2} \ . \end{aligned} \tag{7}$$

Now we find the eigenstates. First make the substitutions

$$\bar{E} \equiv \frac{H_{11}+H_{22}}{2} \text{ and} \tag{8}$$

$$\Delta E \equiv \frac{H_{11}-H_{22}}{2}, \tag{9}$$

so that $H_{11} = \bar{E} + \Delta E$ and $H_{22} = \bar{E} - \Delta E$. The matrix equation (4) becomes

$$\begin{pmatrix} \bar{E}+\Delta E & H_{12} \\ H_{21} & \bar{E}-\Delta E \end{pmatrix} \begin{pmatrix} \alpha_j \\ \beta_j \end{pmatrix} = E_j \begin{pmatrix} \alpha_j \\ \beta_j \end{pmatrix} \tag{10}$$

where

$$E_j = \bar{E} \pm \sqrt{\Delta E^2 + |H_{12}|^2} \tag{11}$$

The ground state energy, corresponding to the minus sign above, will be labelled E_1, with eigenstate coefficients α_1 and β_1; E_2 shall be the energy corresponding to the plus sign with coefficients α_2 and β_2. To get the coefficients α and β, we use one of the equations obtained from the matrix equation (10), plus the normalization requirement. We can either multiply the first row of the matrix by the column vector and set it equal to the energy times α, or multiply the second row by the column vector and set it equal to the energy times β. Either way we would get exactly the same equation.

Choosing the top row and solving for the $j = 1$ state, we have

$$\begin{aligned} (\bar{E}+\Delta E)\alpha_1 + H_{12}\beta_1 &= (\bar{E} - \sqrt{\Delta E^2 + |H_{12}|^2})\alpha_1 \ , \\ \Delta E \alpha_1 + H_{12}\beta_1 &= -\sqrt{\Delta E^2 + |H_{12}|^2}\alpha_1 \\ \beta_1 &= \frac{1}{H_{12}}\left(-\sqrt{\Delta E^2 + |H_{12}|^2} - \Delta E\right)\alpha_1 \ . \end{aligned} \tag{12}$$

The normalization requirement gives us

$$|\alpha_1|^2 + |\beta_1|^2 = 1 \ , \tag{13}$$

$$|\alpha_1|^2 + \frac{1}{|H_{12}|^2}\left(-\sqrt{\Delta E^2 + |H_{12}|^2} - \Delta E\right)^2 |\alpha_1|^2 = 1 \ ; \tag{14}$$

$$|\alpha_1|^2 = \frac{|H_{12}|^2}{|H_{12}|^2 + \left(\sqrt{\Delta E^2 + |H_{12}|^2} + \Delta E\right)^2} \ , \tag{15}$$

$$|\beta_1|^2 = \frac{\left(\sqrt{\Delta E^2 + |H_{12}|^2} + \Delta E\right)^2}{|H_{12}|^2 + \left(\sqrt{\Delta E^2 + |H_{12}|^2} + \Delta E\right)^2} \ . \tag{16}$$

The algebra for the $j = 2$ states is nearly identical; we obtain the conditions

$$\beta_2 = \frac{1}{H_{12}}\left(\sqrt{\Delta E^2 + |H_{12}|^2} - \Delta E\right)\alpha_2 \ , \tag{17}$$

$$|\alpha_2|^2 = \frac{|H_{12}|^2}{|H_{12}|^2 + \left(\sqrt{\Delta E^2 + |H_{12}|^2} - \Delta E\right)^2} , \tag{18}$$

$$|\beta_2|^2 = \frac{\left(\sqrt{\Delta E^2 + |H_{12}|^2} - \Delta E\right)^2}{|H_{12}|^2 + \left(\sqrt{\Delta E^2 + |H_{12}|^2} - \Delta E\right)^2} . \tag{19}$$

Equations (12, 15-19) tell us all we need to specify the eigenstates. A state multiplied by a phase factor $e^{i\phi}$ remains an eigenstate, and also remains normalized. It therefore suffices for each eigenstate to specify the norm of each coefficient, as well as their relative phase. Eqs.(12) and (17) specify the relative phase between α_j and β_j for $j = 1,\ 2$, respectively. If we let

$$H_{12} = re^{i\theta} , \tag{20}$$

and choose β_1 and β_2 to be real, then

$$\alpha_j = e^{i\theta}\ |\alpha_j| \tag{21}$$

since the square root and ΔE appearing in (12) and (17) are real. Thus the eigenproblem is completely solved.

The derivation above can be summarized as follows: to the hamiltonian (1) correspond two stationary states $|\tilde{1}\rangle$, $|\tilde{2}\rangle$ together with energies E_1 (the lower energy),and E_2:

$$|\tilde{1}\rangle = \begin{pmatrix}\alpha_1\\ \beta_1\end{pmatrix} = \frac{1}{\left[|H_{12}|^2 + \left(\sqrt{\Delta E^2 + |H_{12}|^2} + \Delta E\right)^2\right]^{1/2}} \begin{pmatrix} -H_{12} \\ \sqrt{\Delta E^2 + |H_{12}|^2} + \Delta E \end{pmatrix}$$

$$|\tilde{2}\rangle = \begin{pmatrix}\alpha_2\\ \beta_2\end{pmatrix} = \frac{1}{\left[|H_{12}|^2 + \left(\sqrt{\Delta E^2 + |H_{12}|^2} - \Delta E\right)^2\right]^{1/2}} \begin{pmatrix} H_{12} \\ \sqrt{\Delta E^2 + |H_{12}|^2} - \Delta E \end{pmatrix} .$$

$$E_1 = \bar{E} - \sqrt{\Delta E^2 + |H_{12}|^2} \tag{22}$$

$$E_2 = \bar{E} + \sqrt{\Delta E^2 + |H_{12}|^2} \tag{23}$$