Problem 1 (10 pts)

We have particles of energy $E > 0$ incident from the left ($x \to +\infty$) on a step at $x = 0$ that drops to the potential $-V_o$, where V_o is positive. Setting this up and solving in the usual way, we have:

$$\psi_1(x) = e^{-ikx} + re^{ikx} \quad x > 0; \tag{1}$$

$$\psi_2(x) = te^{-iqx} \quad x < 0. \tag{2}$$

Here k and q are defined through

$$E = \frac{\hbar^2 k^2}{2m}, \tag{3}$$

$$E + V_o = \frac{\hbar^2 q^2}{2m} . \tag{4}$$

The matching conditions, from the fact that $\psi(x)$ must be continuous up to first derivative, are

$$t = 1 + r \text{ and} \tag{5}$$

$$-iqt = ik(r-1) . \tag{6}$$

Solving, we get

$$r = \frac{k-q}{k+q} \qquad \text{and} \tag{7}$$

$$t = 1 + r = \frac{2k}{k+q} . \tag{8}$$

The transmission coefficient is defined as the ratio of the transmitted flux to the incident flux. The reflection coefficient is the ratio of the reflected flux to the incident flux. In this case

$$R = |r|^2 = \frac{2E + V_o - 2\sqrt{E(E+V_o)}}{2E + V_o - 2\sqrt{E(E+V_o)}} . \tag{9}$$

The transmission coefficient T could be obtained from $T = 1 - R$. If one wants to obtain it from the amplitude $\psi_2(x)$, however, one must be careful to multiply by a factor of q/k, because the wave vectors are not the same in the two regions. The flux in region 2 is moving faster than that of region 1 by the factor of q/k. However you calculate it,

$$T = \frac{q}{k}|t|^2 = 1 - R = \frac{4\sqrt{E(E+V_o)}}{2E + V_o - 2\sqrt{E(E+V_o)}} . \tag{10}$$

Problem 2

We want to find the momemtum-space representation of the linear harmonic oscillator. It turns out that this can be done using nothing more than a change of variable. There are two ways to see what this change of variable might be: looking at the hamiltonian, or at the raising and lowering operators. I'll do both here.

Using the fact that $\hat{x} = i\hbar\partial_p$ in the momentum representation, and that $\hat{p} = -i\hbar\partial_x$ in the position representation, we have

$$\begin{aligned} \mathrm{x-rep}: \hat{H} &= -\frac{\hbar^2}{2m}\partial_x^2 + \frac{1}{2}m\omega^2 x^2 \text{ and} && (11)\\ \mathrm{p-rep}: \hat{H} &= -\frac{\hbar^2}{2}m\omega^2\partial_p^2 + \frac{1}{2m}p^2 \ . && (12) \end{aligned}$$

Now forget for a moment what p actually represents. If you simply started from the position representation and made the change of variable from x to p defined by

$$p^2/2m \equiv \frac{1}{2}m\omega^2 x^2 \tag{13}$$

or

$$x = p/m\omega \ , \ \partial_x = \partial_p \frac{\partial p}{\partial x} = m\omega\partial_p \ , \tag{14}$$

we obtain an expression that is exactly the same as the momentum representation of the hamiltonian! Notice that this was done *without* using the momentum representation of the $\hat{x}$ operator.

Now the linear harmonic oscillator wave functions are given by

$$\phi_n(x) = \left(\frac{m\omega}{\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} e^{-m\omega x^2/2\hbar} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right) \ . \tag{15}$$

Since the new hamiltonian resulted from a change of variable, we make the substitions and conclude that $\tilde{\phi}_n(p)$ is proportional to

$$\tilde{\phi}_n(p) \propto e^{-p^2/2m\hbar\omega} H_n\left(\frac{1}{\sqrt{m\hbar\omega}}p\right) \ . \tag{16}$$

We can figure out what the proportionality constant is by looking at the normalization. We know that the two representations of the wave function have the same normalization. We can use the change of variable to rewrite the normalization condition as

$$\int_{-\infty}^{\infty} dx \phi_n(x) = \int_{-\infty}^{\infty} \frac{1}{m\omega} dp \tilde{\phi}_n(p) \ . \tag{17}$$

Notice that if we now just insert a $(1/m\omega)^{1/2}$ factor into the normalization of the momentum wave function, we will have the correct normalization. The *almost correct* answer is then

$$\tilde{\phi}_n(p) = \left(\frac{1}{m\hbar\omega}\right)^{1/4} \frac{1}{\sqrt{2^n n!\sqrt{\pi}}} e^{-p^2/2m\hbar\omega} H_n\left(\frac{1}{\sqrt{m\hbar\omega}}p\right) \quad (\textit{almost right!}) \; . \tag{18}$$

We can see that this result can't be quite right by the following reasoning. The momentum space wave functions are still defined as the Fourier transforms of the position space wave functions. However, the Fourier transform of an even function is always purely real (because only the even cosine part of the complex exponential remains) while the transform of an odd function is purely imaginary. But all the wave functions in Eq. 18 are real. Therefore the odd-numbered wave functions can't be the Fourier transforms of their position-space counterparts.

We can see what went wrong by looking at the two different representations of the lowering operators. We have

$$\mathrm{x} : \hat{a}^- = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}x + \sqrt{\frac{\hbar}{m\omega}}\partial_x\right) \quad ; \tag{19}$$

$$\mathrm{p} : \hat{a}^- = \frac{1}{\sqrt{2}}\left(i\sqrt{\frac{1}{m\hbar\omega}}p + i\sqrt{m\omega\hbar}\partial_p\right) \quad . \tag{20}$$

We see from this that the correct substitution to make is

$$x = \frac{ip}{m\omega} \; . \tag{21}$$

The lowering operator has an extra factor of i, and the raising operator can be shown to have an extra factor of $-i$. These cancel out when we form the hamiltonian, since $\hat{H} = \hat{a}^+\hat{a}^- + \frac{1}{2}$, so we didn't see the factor of i when we started with the hamiltonian.

Thus, the correct momentum wave functions are proportional to

$$H_n\left(\frac{i}{\sqrt{m\hbar\omega}}p\right) \; . \tag{22}$$

Problem 3

We want to find when three bound states exist for the finite square well. We let the width be $2a$ and the depth of the well be $-V_o$. Since the ground state is even, and the states alternate even, odd, as one goes up in energy, the third highest bound state will be even. We know that there is always at least one even state, but if the well is too shallow or narrow, there might not be any others. Thus, we need to find a condition on V_o and a so that the second even bound state exists.

As found in problem 4 of Problem Set 6, the condition on the energies of the states with even symmetry is

$$tan\ ka = q/k\ . \tag{23}$$

Here q and k are defined through

$$-E = \frac{\hbar^2 q^2}{2m} \text{ and} \tag{24}$$

$$E + V_o = \frac{\hbar^2 k^2}{2m}\ . \tag{25}$$

If we define

$$c^2 \equiv \frac{2mV_o a^2}{\hbar^2}\ , \tag{26}$$

then $k^2a^2 = c^2 - q^2a^2$, so we have the condition

$$tan\ ka = \frac{\sqrt{c^2 - (ka)^2}}{ka}\ . \tag{27}$$

Since the right-hand-side of the the above equation goes to zero at $ka = c$, the condition for the existence of at least two even states, i.e. three total bound states, is

$$c > \pi\ , \text{ or} \tag{28}$$

$$\frac{2mV_oa^2}{\hbar^2} > \pi^2\ . \tag{29}$$

Problem 4 (20 pts)

We have a particle in a potential corresponding to a linear harmonic oscillator for time $t < 0$ and a potential corresponding to a linear harmonic oscillator plus a force F_o for $t > 0$. The new potential for $t > 0$ is $V(x) = \frac{1}{2}m\omega^2x^2 - F_ox$.

(a) If you sketch the potential, you should find that $V(x)$ is a parabola with its vertex at $(x, V(x)) = (0, 0)$ for $t < 0$, and a parabola with the same curvature but vertex shifted down and to the right. We know the curvature is the same because the x^2 coefficient hasn't changed. We therefore might suspect from this sketch that the displaced system will have the same wave functions, just shifted right by the same amount as the potential, and the same spacing between the energy levels, but shifted downward, again by the same amount as the potential.

(b) To demonstrate this, we rewrite the hamiltonian in the following form:

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 - F_ox \tag{30}$$

$$= \frac{p^2}{2m} + \frac{1}{2}m\omega^2 \left(x^2 - \frac{2F_o}{m\omega^2}x + \left(\frac{F_o}{m\omega^2}\right)^2 \right) - \left(\frac{F_o}{m\omega^2}\right)^2 \cdot \frac{1}{2}m\omega^2 \quad (31)$$

$$= \frac{p^2}{2m} + \frac{1}{2}m\omega^2 \left(x - \frac{F_o}{m\omega^2}\right)^2 - \frac{F_o^2}{2m\omega^2} \ . \quad (32)$$

This hamiltonian has the form $p^2/2m + \frac{1}{2}m\omega^2\chi^2$, where $\chi \equiv x - F_o/m\omega^2$. Therefore the wave functions corresponding to the stationary states of this potential are

$$\psi_n^F(x) = \psi_n(\chi) = \psi_n(x-a) \ ; \ a = \frac{F_o}{m\omega^2} \ , \quad (33)$$

where $\psi_n(x)$ correspond to the stationary state wave functions of the undisplaced hamiltonian. Also, the energies will all be shifted by the constant factor that appears at the end of the hamiltonian:

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right) + \Delta E \ ; \ \Delta E = -\frac{F_o^2}{2m\omega^2} \ . \quad (34)$$

(c) The complete time dependent wave function can be written in the form

$$\psi(x,t) = \sum_{n=0}^{\infty} c_n(0) e^{-iE_nt/\hbar}\psi_n(x-a) \quad (35)$$

where E_n and a are as given in (b). At $t = 0$, we have

$$\psi(x,0) = \psi_0(x,0) = \sum_{n=0}^{\infty} c_n(0)\psi_n(x-a) \ , \text{ or} \quad (36)$$

$$\psi_0(x+a) = \sum_{n=0}^{\infty} c_n(0)\psi_n(x) \ . \quad (37)$$

Now we are almost in a position where we can use the formulae given on the test sheet. We notice first of all the that the stationary state wave functions $\psi_n(x)$ are not exactly the same as the functions $\phi_n(x)$ given in the formulae, but are in fact related by a scale factor:

$$\psi_n(x) = \phi_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right) \ . \quad (38)$$

It follows that

$$\psi_n(x+a) = \phi_n\left(\sqrt{\frac{m\omega}{\hbar}}(x+a)\right) = \phi_n\left(\sqrt{\frac{m\omega}{\hbar}}x + \sqrt{\frac{m\omega}{\hbar}}a\right) \ . \quad (39)$$

Hence we have

$$\phi_0\left(\sqrt{\frac{m\omega}{\hbar}}x + \sqrt{\frac{m\omega}{\hbar}}a\right) = \sum_{n=0}^{\infty} c_n(0)\phi_n(\sqrt{\frac{m\omega}{\hbar}}x) \text{ , or} \tag{40}$$

$$\phi_0(y+b) = \sum_{n=0}^{\infty} c_n(0)\phi_n(y) \text{ ,} \tag{41}$$

where the substitutions $y \equiv \sqrt{\frac{m\omega}{\hbar}}x$ and $b = \sqrt{\frac{m\omega}{\hbar}}a$ were made. Now we can immediately see from the formula that

$$c_n(0) = exp\left(-b^2/4\right)\frac{(-b)^n}{\sqrt{2^n n!}} \text{ ,} \tag{42}$$

so that

$$c_n(t) = exp\left(-\frac{m\omega a^2}{4\hbar}\right)\frac{(-\sqrt{\frac{m\omega}{\hbar}}a)^n}{\sqrt{2^n n!}} exp\left(\frac{-it}{\hbar}\left(\hbar\omega\left(n+\frac{1}{2}\right) - \frac{F_o^2}{2m\omega^2}\right)\right) \text{ .} \tag{43}$$

(d) As a check we can make sure that the sum of the squared norms of the $c_n(t)$ is 1. To do this we needed to know the power series expansion

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ .} \tag{44}$$

Using the fact that $|c_n(t)|^2 = |c_n(0)|^2$ since the time dependence is contained in a simple phase factor, we have

$$\sum_{n=0}^{\infty} |c_n(t)|^2 = e^{-b^2/2}\sum_{n=0}^{\infty}\frac{(-b)^{2n}}{2^n n!} \tag{45}$$

$$= e^{-b^2/2}\sum_{n=0}^{\infty}\left(b^2/2\right)^n/n! \tag{46}$$

$$= e^{-b^2/2}e^{+b^2/2} \tag{47}$$

$$= 1 \text{ .} \tag{48}$$

The set of stationary states of the displace oscillator form a complete orthonormal basis just like the old one. Since the quantities $|c_n(t)|^2$ correspond to the probability of finding the particle in the n^{th} displaced stationary state at time t, and since the particle must be found in one of these states, the sum of the probabilities must be 1.