Problem 1: Goswami, Chapter 6, Problem 4

Operators are hermitian if they are equal to their adjoint. To form the hermitian adjoint of an expression, do the following:

- Change all bras into kets, and all kets into bras.
- Complex conjugate all numbers.
- Turn all operators into their adjoints; if the operators are hermitian you don't have to do anything to them.
- If you have a product of operators, or a product of bras, operators and kets such as $\langle\psi|\,\hat{A}\hat{B}\,|\psi'\rangle$, taking the hermitian adjoint involves *reversing the order* of everything. Thus the hermitian adjoint of the above expression is $\langle\psi'|\,\hat{B}^\dagger\hat{A}^\dagger\,|\psi\rangle$.

Therefore, for any operator A, $(A+A^\dagger)^\dagger = A^\dagger + (A^\dagger)^\dagger = A^\dagger + A$, and $[i(A-A^\dagger)]^\dagger = -i(A^\dagger - A) = i(A - A^\dagger)$.

Problem 2: Goswami, Chapter 6, Problem 5

The expectation value of A^2 can be written in Dirac notation as $\langle A^2\rangle = \langle\psi|\,A^2\,|\psi\rangle$, where $|\psi\rangle$ is an *arbitrary* state vector. The operator A is assumed to be hermitian, so it has a complete orthornormal basis $\{|\phi_n\rangle\}$. By inserting a resolution of the identity in this basis, we obtain

$$\begin{aligned}\langle\psi|\,A^2\,|\psi\rangle &= \sum_n \langle\psi|\,A^2\,|\phi_n\rangle\,\langle\phi_n|\psi\rangle \\ &= \sum_n \lambda_n^2\,\langle\psi|\phi_n\rangle\,\langle\phi_n|\psi\rangle \\ &= \sum_n \lambda_n^2\,|c_n|^2\end{aligned}$$

which is the sum of non-negative numbers since the eigenvalues λ of A are real. Therefore the expectation value of A^2 is positive for any state $|\psi\rangle$.

Problem 3: Goswami, Chapter 6, Problem A7

(a) H_1 and H_2 are hermitian as a direct application of the results of problem 1.

(b) Evaluating the commutators is simplified by application of the formulae $[A+B,C]=[A,C]+[B,C]$ and $[A,C+D]=[A,C]+[A,D]$; in other words, commutation is distributive. Also, the commutator of any operator with itself is zero. Finally, you can factor out any constants.

$$\begin{aligned}[H_1,H_2] &= [\frac{U+U^\dagger}{2},\frac{U-U^\dagger}{2}] &(1)\\ &= \frac{1}{4i}\left([U,U]-[U,U^\dagger]+[U^\dagger,U]-[U^\dagger,U^\dagger]\right) &(2)\\ &= \frac{1}{4i}\left(0-(1-1)+(1-1)-0\right) &(3)\\ &= 0. &(4)\end{aligned}$$

We used the fact that U was unitary. We also showed that the commutators of U with itself and with its adjoint $U^\dagger$ are both zero, so $[U,H_1]$ and $[U,U_2]$ must be zero also.

(c) Since U, H_1, and H_2 all commute with each other they have simultaneous eigenstates. Eigenstates are typically labeled by their eigenvalues, or by some numbers that are related to the eigenvalues. We call these numbers the *quantum numbers* that label the states. We always label the state by the eigenvalues of *hermitian* operators, since then the eigenvalues are real.

The state $|h_1h_2\rangle$ is precisely that state whose eigenvalue of H_1 is h_1 and whose eigenvalue of H_2 is h_2. It be an eigenstate of H_1 and H_2 at the same time because H_1 and H_2 commute with each other. The state $|h_1h_2\rangle$ is evidently also an eigenstate of U, with eigenvalue h_1+ih_2. Therefore, taking the hermitian adjoint (see Problem 1) we obtain

$$
\begin{aligned}
\hat{U}\,|h_1h_2\rangle &= (h_1 + ih_2)\,|h_1h_2\rangle \ , \\
\langle h_1h_2|\,\hat{U}^\dagger &= \langle h_1h_2|\,(h_1 - ih_2) \ , \\
\langle h_1h_2|\,\hat{U}^\dagger U\,|h_1h_2\rangle &= \langle h_1h_2|h_1h_2\rangle \\
&= \langle h_1h_2|\,(h_1 - ih_2)(h_1 + ih_2)\,|h_1h_2\rangle \ \text{sothat} \\
(h_1 - ih_2)(h_1 + ih_2) = |h_1 + ih_2|^2 = 1.
\end{aligned}
$$

Problem 4: Goswami, Chapter 7, Problems A8 and A9

(a) $(c^\dagger c)^\dagger = c^\dagger c$, so N is hermitian.

(b) $N^2 = c^\dagger c\; c^\dagger c = c^\dagger(1 - c^\dagger c)c = N - (c^\dagger)^2 c^2 = N$.

(c) If $|n\rangle$ is an eigenstate of N with eigenvalue n, then

$$
\begin{aligned}
N^2\,|n\rangle &= n^2\,|n\rangle \\
&= N\,|n\rangle = n\,|n\rangle \ \text{ so that} \\
n^2 &= n \ ,
\end{aligned}
$$

which implies that $n = 0$ or 1.

(d) To show that $c\,|0\rangle = 0$ notice that its inner product with itself is $\langle 0|\,c^\dagger c\,|0\rangle = \langle 0|\,\hat{N}\,|0\rangle = 0$. Therefore the state $c\,|0\rangle$ has zero modulus, so it must be equal to zero. For the other part of the question, to show that $c^\dagger\,|0\rangle$ is equal to $|1\rangle$, it suffices to show that when we operate on it with N, we get the same state back again:

$$
\begin{aligned}
N\;c^\dagger\,|0\rangle &= c^\dagger c\; c^\dagger\,|0\rangle \\
&= c^\dagger(1 - c^\dagger c)\,|0\rangle \\
&= c^\dagger\,|0\rangle - (c^\dagger)^2 c\,|0\rangle \\
&= c^\dagger\,|0\rangle \ .
\end{aligned}
$$

Finally, since the hamiltonian can be written as $\hat{H} = \hbar\omega(\hat{N} + \frac{1}{2})$, we can use the fact that $\hat{N}^2 = \hat{N}$ to get

$$
\begin{aligned}
\hat{H}\,|n\rangle &= \hbar\omega(n + \frac{1}{2})\,|n\rangle \\
&= \hbar\omega(n^2 + \frac{1}{2})\,|n\rangle \ \ ; \text{ subtracting gives} \\
n^2\,|n\rangle &= n\,|n\rangle \ .
\end{aligned}
$$

So long as $|n\rangle$ is nonvanishing, we can dot each side of the equation with $\langle n|$ and then divide by $\langle n|n\rangle$ to get $n^2 = n$, which implies that $n = 0$ or 1 as before. A vanisihing state, i.e. a state with zero modulus, is trivially an eigenstate of any hamiltonian whatsoever, with eigenvalue zero, but we're normally only interested in *normalizable* state vectors.