Part I
Problem Set 2 Solutions

1 Problem 1: Normalization and Expectation Values

We had to normalize the wave function

$$\phi(x) = \frac{c}{a^2 + x^2} \tag{1}$$

and find the expectation values $\langle x \rangle$ and $\langle x^2 \rangle$. In order to normalize one of course has to know over what range the wave function is defined; in this case we expected everyone to normalize over the entire x-axis, i.e. from $-\infty$ to $+infty$. Some students normalized over the range $[0, a]$, which, for the purposes of this problem was acceptable.

When we say *normalize the wave function*, we mean, multiply the wave function by some constant (which could be complex) so that, when you integrate the norm of the whole thing squared, you get one. In this problem, c is the normalization constant you are solving for. This constant is found by solving

$$\begin{aligned}
\int_{-\infty}^{+\infty} |\phi(x)|^2 \; dx &= \int_{-\infty}^{+\infty} |c|^2 \left(\frac{1}{a^2 + x^2}\right)^2 \; dx && (2)\\
&= |c|^2 \int_{-\infty}^{+\infty} \left(\frac{1}{a^2 + x^2}\right)^2 \; dx && (3)\\
&= 1 && (4)
\end{aligned}$$

for c. Many students looked the integral up in a table, but it really isn't that hard; if you substitute $x = a \; tan(\theta)$, the integrand becomes $cos^2(\theta)$.

Answer: $c = \sqrt{2a^3/\pi}$.

The expectation values you need to compute are

$$\langle x \rangle = \int_{-\infty}^{+\infty} dx \frac{c^2}{(a^2 + x^2)^2} x \tag{5}$$

and

$$\langle x^2 \rangle = \int_{-\infty}^{+\infty} dx \frac{c^2}{(a^2 + x^2)^2} x^2 \tag{6}$$

Both of these integrals are also straightforward with the substitution $x = a\ tan(\theta)$ (*but watch the bounds of integration!*). The expectation value of x, though, is easily seen to equal 0 since the integrand is odd.

Answer: $\langle x \rangle = 0$, $\langle x^2 \rangle = a^2$.

2 Problem 2: Particle in a box

For a particle in a stationary state of a box, we want to find the probability of observing the particle in the right side ($x \geq a/2$).

Since we have the general form of the wave function for the stationary states, all we have to do compute

$$\int_{a/2}^{a} dx\ |\phi_n(x)|^2 = \frac{2}{a} \int_{a/2}^{a} dx\ sin^2\ \frac{n\pi x}{a} \tag{7}$$

which equals 1/2 for all n, as you would expect since the wave functions are all either symmetric or antisymmetric about the midpoint of the box.

3 Problem 3: Interaction of an Electron in a Box with Light

We want to find the size of a box which satisfies the condition that the difference in energy between the two lowest states equals the energy of either red light with wavelength $\lambda = 8000\mathring{A}$, or blue light with wavelength $\lambda = 4000\mathring{A}$.

Notice that we are em not finding the box whose lowest energy state *equals* the energy of the incoming light. Transitions take place only between states, so we have to look at the difference in energies.

The energy E_n of the nth stationary state of a particle of mass m in a box of size a is

$$E_n = \frac{\hbar^2}{2m}\frac{\pi^2}{a^2}n^2 \tag{8}$$

The difference in energy between the two lowest states, which we shall call ΔE, is then

$$\Delta E = E_2 - E_1 = \frac{\hbar^2}{2m}\frac{\pi^2}{a^2}(2^2 - 1^2). \tag{9}$$

The energy of the light is given by

$$E_\gamma = \hbar\omega = \frac{2\pi\hbar c}{\lambda}. \tag{10}$$

Setting ΔE equal to E_γ and solving for a^2 gives

$$a^2 = \frac{3\pi}{4}\frac{\hbar c}{mc^2}\lambda. \tag{11}$$

At this point all that is left is to plug in numbers. Quite a few people got tripped up at this point by using inconsistent units; the eV/c^2 units of mass got a number of students. I find it helpful, when I have to plug in numbers for the physical constants, to put them in standard combinations like $\hbar c$ and mc^2. Using the values

$$\hbar c = 1973 eV - \mathring{A} \tag{12}$$

$$mc^2 = 511 \times 10^3\ eV \tag{13}$$

we get

$$a = 0.095\sqrt{\lambda}\ \mathring{A} \tag{14}$$

when λ is also expressed in Angstroms.

: Answers:

$$\lambda = 8000\ \mathring{A} : \qquad a = 8.53\ \mathring{A} \tag{15}$$

$$\lambda = 4000\ \mathring{A} : \qquad a = 6.03\ \mathring{A} \tag{16}$$

4 Problem 4: Wave Packet in One-Dimensional Box - Difficult