Part I
Problem Set 1 Solutions

1 Problem 2

(a) Use Planck's radiation frequency distribution law to obtain the energy density as a distribution in wave length λ. Determine the wave length λ_{max} at which the energy density assumes its maximum value. Demonstrate the relationship $\lambda_{max} = b/T$ and determine the value of the constant b.

To do part (a) you had to start with an expression for the spectral energy density, which means the amount of radiating energy at a particular frequency you would find coming out of a black body. In terms of the angular frequency ω, this can be written

$$u_\omega = \frac{1}{\pi^2 c^3} \frac{\hbar\omega^3}{e^{\hbar\omega/k_B T} - 1} d\omega. \tag{1}$$

This means the amount of energy found in the spectrum between ω and $\omega + d\omega$. Using the change of variable $\omega = 2\pi c/\lambda$, you can derive the energy density as a function of wavelength λ.

There are two algebraic points that many people missed. Both can be understood by realized that the energy density is defined by the expression

$$\begin{aligned} U &= \int_0^\infty u_\omega d\omega \qquad (2) \\ &= \int_0^\infty u_\lambda d\lambda \qquad (3) \end{aligned}$$

To go from the first equation to the second, you have to include $d\omega = -2\pi c/\lambda^2 d\lambda$; otherwise your new expression for the total energy would have a $d\omega$ in it instead of a $d\lambda$, which is nonsensical.

Here's another way to look at it. Say you simply took the expression for the energy density $u(\omega)$ made the change of variable from ω to λ, and called the result $\tilde{u}(\lambda)$. That's fine, except that now $\tilde{u}(\lambda)$ no longer corresponds to the energy density! That's because the total energy would now be

$$U = \int_0^\infty d\lambda \, \frac{2\pi c}{\lambda^2} \, \tilde{u}(\lambda). \tag{4}$$

Some of you also were confused about the $-$ sign; notice that when we make the change of variable the bounds of integration are interchanged, so that flipping them back cancels the $-$ sign introduced by the $d\omega$.

Many students began with an expression for the energy density that was off by a factor of two. Apparently this came from class notes and so no points were deducted for using (we'll try to catch mistakes like that sooner in the future!). The factor of two comes from the fact that an electromagnetic has two possible polarizations.

To find λ_{max} as a function of temperature, almost everyone made the approximation, valid in this case that

$$e^{2\pi\hbar c/\lambda k_B T} \gg 1. \tag{5}$$

However, hardly anybody made any attempt to justify this approximation! It's always important to make sure that these approximations lead to consistent results. The solution is straightforward once the approximation is made, but the problem could also be solved numerically, by first making a change of variable to put the equation in dimensionless form, the solving it using MATLAB, Mathematica, or your pocket calculator.

Answers:

$$u_\lambda = \frac{\hbar c}{\lambda^5} 16\pi^2 \left[exp\left(\frac{2\pi\hbar c}{\lambda k_B T}\right) - 1\right]^{-1} \tag{6}$$

$$b = 2.909x10^{-3} m - K \tag{7}$$

2 Problem 3: De Broglie wave length

(a) Calculate the de Broglie wave length of a 10 eV and of a 10 MeV electron.

The energy of a particle is always given by the relativistic energy-momentum relationship

$$E^2 = p^2c^2 + m^2c^4, \tag{8}$$

where E is the total energy of the particle, p it's relativistic momentum and m its rest mass. At low speeds this expression reduces to the usual formula for the kinetic energy: $E = p^2/2m$. At high velocities, the momentum term dominates, so that $E \approx pc$. The question is, when are the velocities (and hence the energies) relativistic? The rest mass of an electron is $.511 MeV/c^2$. When its energy is much less than $1MeV$, the non-relativistic energy-momentum relation $E = p^2/2m$ is appropriate. But for energies on the order of or greater than $1MeV$, the relativistic expression is more accurate.

Answers (don't worry if yours are different by a few percent):

$$E = 10eV \qquad \lambda = 3.88\mathring{A} \tag{9}$$
$$E = 10MeV \qquad \lambda = 1.24 \times 10^{-3}\mathring{A} \tag{10}$$
$$\lambda = 1\mathring{A} \qquad E = 150MeV \tag{11}$$
$$\lambda = 10^{-17}cm \qquad E = 1.24 \times 10^{13}eV \tag{12}$$

3 Problem 4: Klein-Gordon Equation

Nobody had any trouble with this problem; the result follows directly from the de Broglie relations.