Problem 1: Two Spin-$\frac{1}{2}$ Particles

(a) Two spin-$\frac{1}{2}$ particles in an external static magnetic field $B_o \hat{e}_3$ are described by the Hamiltonian

\[
H = J \vec{S}^{(1)} \cdot \vec{S}^{(2)} + g \mu_B B_o \left( S_3^{(1)} + S_3^{(2)} \right)
\]  

(1)

where $g$ and $\mu_B$ are well-known physical constants. Determine the stationary states and the respective energies of the system. Sketch the energies as a function of $B_o$.

(b) Apply a weak, time-dependent magnetic field $B(t) = b_o \cos(\omega t) \hat{e}_1$. What values of $\omega$ must be chosen to induce transitions between the eigenstates in (a).

(c) For properly chosen $\omega$ determine the transition rates in leading order perturbation theory.

Problem 2: Selection Rules for One-Photon Absorption in Hydrogen Atoms

Determine the selection rules for one-photon absorption processes in the hydrogen atom, i.e., for which combination of quantum numbers $n, \ell, m$ for the initial state and $n', \ell', m'$ for the final state one can expect non-zero absorption rates. Express for this purpose the operator $\vec{r}$ in the transition dipole moment through spherical tensor operators and employ the Wigner-Eckart theorem.

Problem 3: Ethylene $\pi$-Electron States

The molecule ethylene $C_2H_4$ has two $2p_z$ electrons on each of its carbons. Denoting the atomic orbitals (including spin) by $|j, \sigma\rangle$, $j = 1, 2$; $\sigma = \pm \frac{1}{2}$ assume that the Hamiltonian for the system is

\[
H = -t \sum_{\sigma} \left( c_{1\sigma}^\dagger c_{2\sigma} + c_{2\sigma}^\dagger c_{1\sigma} \right) + U \left( c_{1\alpha}^\dagger c_{1\alpha} c_{1\beta}^\dagger c_{1\beta} + c_{2\alpha}^\dagger c_{2\alpha} c_{2\beta}^\dagger c_{2\beta} \right).
\]  

(2)

Here $\alpha, \beta$ denote spin-up ($|\frac{1}{2}, \frac{1}{2}\rangle$) and spin-down ($|\frac{1}{2}, -\frac{1}{2}\rangle$) states, respectively.

(a) State all possible states of the system in the given basis of single electron states.
(b) Determine the Hamiltonian matrix in the basis of all 2-electron states.

(c) State all triplet states and their energies.

(d) Determine the energy of the lowest singlet state (ground state) in 2nd order (with respect to $U$) perturbation theory. Note: In second order perturbation theory holds for the energy of a state $|0\rangle$ (in the usual notation)

$$E_0 \approx \text{zero order} + \text{first order contributions} + \sum_{n,n\neq 0} \frac{\langle 0|V|n\rangle\langle n|V|0\rangle}{\epsilon_0 - \epsilon_n}. \quad (3)$$

All material, e.g., books, class notes, are allowed in the exam, except solutions (your own or those of others) of the homework sets of this class.