Problem 1: Lie Algebra of the Group SU(3)

Consider the commutation properties of the eight generators of the group SU(3)

\[ \lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]

\[ \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \]

\[ \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} , \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \]

\[ \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} , \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \]

Using Mathematica verify the commutation properties of these matrices

\[ [\lambda_j, \lambda_k] = 2i \delta_{j\ell} \lambda_\ell \]  
(1)

\[ [\lambda_j, \lambda_k] + = \frac{4}{3} \delta_{jk} \mathbb{I} + 2d_{j\ell} \lambda_\ell . \]  
(2)

Employ the matrix notation of Mathematica, e.g., for the matrix \( \lambda_1 \) above

\[ l1 = \{\{0,1,0\},\{1,0,0\},\{0,0,0\}\} \]

(3)

and use the built-in matrix product ".", e.g., “l1.l2”. Also note that matrix elements are addressed in Mathematica like l1[[i,i]]. Furthermore, to evaluate the trace of a matrix use a command like Sum[ l1[[i,i]], \{i,1,3\}].

(a) Verify that the following property holds

\[ \text{tr} (\lambda_j \lambda_k) = 2 \delta_{jk} . \]  
(4)
(b) Using (1, 2) demonstrate that $f_{jk\ell}$ is a totally anti-symmetric tensor, i.e., any transposition of two indices yields the same value, except for a change of sign.

(c) Using (1, 4) demonstrate that $d_{jk\ell}$ is a totally symmetric tensor, i.e., for any transposition of two indices holds $d_{jk\ell} = d_{kj\ell}$ etc.

(d) Verify the table of non-vanishing $f_{jk\ell}$ and $d_{jk\ell}$ given in class. For this purpose employ the compact expression of these constants introduced in (b,c).

(e) Demonstrate that the following two matrices (so-called Casimir operators) commute with all generators $\lambda_j$:

$$C_1 = \sum_{j=1}^{8} \lambda_j^2$$
$$C_2 = \sum_{j,k,\ell} d_{jk\ell} \lambda_j \lambda_k \lambda_\ell . \quad (5)$$

Problem 2: Matrix Representations of Quark and Anti-Quark Multiplets

(a) Consider the SU(3) multiplet $[3]$ describing the family of "u", "d", and "s" quarks. Express the action of the operators $T_{\pm,3}$, $U_{\pm,3}$, $V_{\pm,3}$ on the states of the quark multiplet through appropriate $3 \times 3$-matrices.

(b) Using the relationships which connect the operators $T_{\pm,3}$, $U_{\pm,3}$, $V_{\pm,3}$ with the operators $F_j$, $j = 1, 2, \ldots, 8$

$$T_\pm = F_1 \pm iF_2 , \quad T_3 = F_3$$
$$V_\pm = F_4 \pm iF_5 , \quad U_{\pm} = F_6 \pm iF_7$$
$$V_3 = \frac{3}{2} Y + \frac{1}{2} T_3 , \quad U_3 = \frac{3}{4} Y - \frac{1}{2} T_3 , \quad Y = \frac{2}{\sqrt{3}} F_8 \quad (6)$$

and $F_j = \frac{1}{2} \lambda_j$ $j = 1, 2, \ldots, 8$ express $\lambda_j$ in terms of $3 \times 3$-matrices.

(c) Repeat (a) and (b) for the multiplet $\bar{3}$ representing the anti-quarks.
Problem 3: Quark Wave Functions of Spin-$\frac{3}{2}$ Baryons

Baryons are formed from three quarks. States composed of three quarks are elements of the representation $[3] \otimes [3] \otimes [3]$ where $[3]$ is the representation $D(1,0)$ of SU(3) corresponding to the quarks $u, d, s$. The quarks also carry spin–$\frac{1}{2}$. The states of $[3] \otimes [3] \otimes [3]$, including the spin–$\frac{1}{2}$ attributes, will be denoted by

$$ u_d s = u(1) |\frac{1}{2}, -\frac{1}{2} \rangle \uparrow d(2) |\frac{1}{2}, \frac{1}{2} \rangle s(3) |\frac{1}{2}, -\frac{1}{2} \rangle , $$
$$ u u \uparrow s = u(1) |\frac{1}{2}, \frac{1}{2} \rangle u(2) |\frac{1}{2}, \frac{1}{2} \rangle s(3) |\frac{1}{2}, -\frac{1}{2} \rangle $$

etc. The representation $[3] \otimes [3] \otimes [3]$ of SU(3) can be expressed in terms of the irreducible representation $D(3,0)=[10]$, two $D(1,1)=[8]$ and a $D(0,0)=[1]$, i.e.,


In the following we will consider first the representation $[10]$, called the baryon decuplet, and then one of the representations $[8]$, called the baryon octet. The first representation contains the $\Delta$ particles, the particle which appears as a resonance in nucleon–pion scattering, the second representation contains the two nucleons, i.e., the proton and the neutron.

The baryon octet $[10]$ contains an isospin quartet ($I = \frac{3}{2}$) of $\Delta$’s, namely, $\Delta^{++}, \Delta^+, \Delta^0, \Delta^-$, an isospin triplet ($I = 1$) of $\Sigma^*$’s, namely, $\Sigma^{++}, \Sigma^0, \Sigma^-$, an isospin doublet ($I = \frac{1}{2}$) of $\Xi^*$’s, namely, $\Xi^0, \Xi^*$, and finally an isospin singlet ($I = 0$), namely, the $\Omega^-$. All these ‘particles’ have spin $\frac{3}{2}$.

(a) The wave function of the $\Delta^{++}$ is

$$ |\Delta^{++} \rangle = u_\uparrow u_\uparrow u_\uparrow . $$

Using the total spin operators

$$ S_{\pm,3} = (1) S_{\pm,3} + (2) S_{\pm,3} + (3) S_{\pm,3} $$

show that the spin state of $|\Delta^{++} \rangle$ is $|\frac{3}{2}, \frac{3}{2} \rangle$.

(b) What are the eigenvalues of the total isospin and total hypercharge operators

$$ T_3 = (1) T_3 + (2) T_3 + (3) T_3 \quad , \quad Y = (1) Y + (2) Y + (3) Y $$
of $|\Delta^{++}\rangle$? Here the operators $^{(j)}O$ denote the operators acting on the $j$-th quark. Show that the quantum numbers are in agreement with the total charge of +2 of the $\Delta^{++}$ by applying the charge operator $\hat{Q} = \frac{1}{2} Y + T_3$.

(c) Applying the operators
\[
\begin{align*}
T_\pm &= (1)T_\pm + (2)T_\pm + (3)T_\pm \\
V_\pm &= (1)V_\pm + (2)V_\pm + (3)V_\pm \\
U_\pm &= (1)U_\pm + (2)U_\pm + (3)U_\pm
\end{align*}
\]

(12)
generate, starting from $|\Delta^{++}\rangle$ as given in (9), all other states of the baryon decouplet. Use for this purpose $(j = 1, 2, 3)$
\[
\begin{align*}
^{(j)}T_- u(j) &= d(j) , \quad ^{(j)}T_+ d(j) &= u(j) \\
^{(j)}V_- u(j) &= s(j) , \quad ^{(j)}V_+ s(j) &= u(j) \\
^{(j)}U_- d(j) &= s(j) , \quad ^{(j)}U_+ s(j) &= d(j)
\end{align*}
\]

(13)
and the fact that all other actions of these operators lead to ‘zero’, e.g.,
\[
^{(j)}V_- d(j) = 0.
\]

(d) Provide the wave function which describes the $\Delta^+$ in the spin state $|\frac{3}{2}, \frac{1}{2}\rangle$.

(e) Using (13) determine the representation of the operators $V_\pm$, $U_\pm$ and $T_\pm$ corresponding to [10]. Note that these operators, in case of representation [10], are 10×10 matrices. You should exercise care in accounting properly for the normalization factors of the states which will modify the respective matrix elements. Provide also the representations of the operators $T_3$ and $Y$ using the quantum numbers $t_3, y$ of all states determined in (c).

(f) Determine the matrices $\lambda_j$, $j = 1, \ldots 8$ corresponding to the operators in (d) employing the relationships $\lambda_1 = T_1 + T_-$, etc. (Use for this tedious job and the remaining part of (f) Mathematica.) Note that the $\lambda_j$’s are also 10 × 10–matrices. Test, if the matrices obey the Lie algebra of the group SU(3), i.e., $[\lambda_j, \lambda_k] = 2i f_{jkl} \lambda_l$. 

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Problem 4: Quark Wave Functions of Spin-\(\frac{1}{2}\) Baryons

(a) Consider now the baryon octet \([8]\). The neutron is an element of this representation with \(t_3, y\) quantum numbers \(-\frac{1}{2}, 1\). The wave function of the neutron in the spin state \(|\frac{1}{2}, \frac{1}{2}\rangle\) is

\[
|n\rangle = \frac{1}{\sqrt{18}} \left( 2d_\uparrow u_\downarrow d_\uparrow + 2d_\uparrow d_\uparrow u_\downarrow + 2u_\downarrow d_\uparrow d_\uparrow + d_\uparrow u_\downarrow d_\uparrow + d_\uparrow d_\uparrow u_\downarrow - u_\downarrow d_\uparrow d_\uparrow - d_\uparrow u_\downarrow d_\uparrow - d_\uparrow d_\uparrow u_\downarrow - u_\downarrow d_\uparrow d_\uparrow \right) .
\]

(14)

Verify that the neutron, thus described, has spin \(\frac{1}{2}\). Construct from the neutron state the state of the proton by applying the operator \(T^+\) as defined in (12).

(b) Construct similarly the wave functions of all states on the perimeter of the \([8]\) multiplet:

\[\begin{array}{c}
\Sigma^- \\
\Xi^0 \\
\Lambda^0 \\
\Xi^0 \\
\Sigma^+ \\
\end{array}\]

Octet of Spin-\(\frac{1}{2}\) Baryons

(c) Verify that the six states constructed in (b) form two \(T\)-doublets, two \(V\)-doublets, and two \(U\)-doublets with respect to the respective total \(T, V, U\)–operators as defined in (12).

(d) Construct now the two central states \(\Sigma^0, \Lambda^0\) proceeding as follows: First construct the state \(\Sigma^0 = |1, 0\rangle_T\) by applying \(T^-\) defined in (12) to the perimeter state \(\Sigma^+ = |1, 1\rangle_T\) (see figure). Then construct the state \(|1, 0\rangle_V\) describing the central state of a \(V\) triplet by applying \(V^-\) to the perimeter state \(p = |1, 1\rangle_V\). Show that the resulting state is not orthogonal to the state \(\Sigma^0 = |1, 0\rangle_T\) constructed already. Obtain a proper orthogonal state.

(e) Show that the state constructed through application of \(U^+\) to the perimeter state \(\Xi^0\) can be expressed in terms of the two states constructed in (d). Show this explicitly by expressing the state \(U^+ \Xi^-\) as a linear combination of
the states constructed in (d). Demonstrate this property also by considering the commutator $[U_+, V_-]$. 

This project needs to be handed in by Friday, May 12, 2000 into the mail box of Gheorghe-Sorin Paraoan in Loomis.