1 Verhulst Equation

(a)  
At the stationary points \( a = x_{1s,2s} \), the time derivative of \( x \) vanishes, i.e.,

\[
\begin{align*}
    x - x^2 &= 0, \\
    x_{1s} &= 0, \\
    x_{2s} &= 1.
\end{align*}
\]

The linear approximation of equation (1, homework) around \( x = a = x_{1s,2s} \) is

\[
\begin{align*}
    \delta \dot{x} &= f(a) + \partial_x f|_a \delta x, \\
    &= (1 - 2a) \delta x.
\end{align*}
\]

Solution for this equation is

\[
\delta x(t) = \delta x(0) e^{(1 - 2a)t}.
\]

and its behavior around the stationary points is

\[
\delta x(t) = \begin{cases} 
    \delta x(0) e^t & x_{1s} = 0 \\
    \delta x(0) e^{-t} & x_{2s} = 1
\end{cases}
\]

In the first case \( x(t) \) moves away from \( x_{1s} = 0 \) and in the second case it gets closer to the \( x_{2s} = 1 \). Accordingly \( x_{1s} \) and \( x_{2s} \) are stable and unstable stationary points, respectively.

(b)  
The exact solution of the equation

\[
\frac{dx}{dt} = x - x^2,
\]

can be derived by writing

\[
\frac{dx}{x - x^2} = dt,
\]

and integrating both sides

\[
\int_{x(0)}^{x(t)} \frac{dx}{x - x^2} = \int_0^t dt.
\]
From this follows

$$\int_{x(0)}^{x(t)} dx \left( \frac{1}{x} + \frac{1}{1-x} \right) = t,$$

(11)

$$\log x - \log(1-x) |_{x_0}^{x(t)} = t,$$

(12)

$$\log \frac{x(t)}{1-x(t)} - \log \frac{x_0}{1-x_0} = t,$$

(13)

$$\frac{x(t)}{1-x(t)} = \frac{x_0}{1-x_0} e^t.$$  

(14)

One can reorganize the equation (14) to obtain the solution for $x(t)$,

$$x(t) = \frac{x_0}{x_0 + (1-x_0)e^{-t}}.$$  

(15)

The behavior of the solution around the stationary points can be obtained by expanding it near stationary points. For $x \approx 1$, one can express $\delta x(t) = x(t) - 1$ from the equation (15),

$$\delta x(t) = \frac{(x_0 - 1)e^{-t}}{x_0 + (1-x_0)e^{-t}}.$$  

(16)

For small values of $\delta x(0) = x_0 - 1$ and $\delta x(t) = x(t) - 1$, equation (16),

$$\delta x(t) = \frac{\delta x(0)e^{-t}}{1+\delta x(0) - \delta x(0)e^{-t}}.$$  

(17)

can be expanded as follows

$$\delta x(t) \approx \delta x(0)e^{-t} \left( 1 - \delta x(0) + \delta x(0)e^{-t} \right) \approx \delta x(0)e^{-t}.$$  

(18)

Similarly one can derive for $x(t) \approx 0$ and $\delta x(0) = x_0 - 0$,

$$\delta x(t) \approx \delta x(0)e^t.$$  

(19)

Equations (18) and (19) are identical to those derived in section (a).

(c)

We can rewrite equation (3, homework) as

$$x_{n+1} = \frac{x_n}{(1-h) + hx_n}.$$  

(20)

From this follows

$$x_{n+1}^{-1} - 1 = (1-h)x_n^{-1} + h - 1,$$

(21)

and

$$x_{n+1}^{-1} - 1 = (1-h)(x_n^{-1} - 1).$$  

(22)
We can now define a new variable, \( u_n = x_n^{-1} - 1 \), so that the recursion relation can be simplified as follows:

\[
u_n = (1 - h)u_{n-1}.
\]

(23)

The solution of this equation is

\[
u_n = (1 - h)^n u_0.
\]

(24)

The answer we seek is obtained by switching back to \( x_n \),

\[
x_n^{-1} = 1 - (1 - x_0^{-1}) (1 - h)^n.
\]

(25)

From this equation one can see that \( x_n \) converges to 1, i.e., \( \lim_{n \to \infty} x_n = 1 \) for \( 1 \geq h > 0 \). \( x_n \) still tends to 1 for \( 2 > h \geq 1 \), but it oscillates around 1, so probably \( h \geq 1 \) is not a good approximation although it still converges to 1.

Replacing \( n \) with \( t/h \) and \( x_n \) with \( x(t) \) in the equation (25) yields

\[
x(t) = \frac{1}{1 - (1 - x_0^{-1})(1 - h)^{t/h}}.
\]

(26)

The term \( (1 - h)^{t/h} \) becomes \( e^{-t} \) in the limit where \( h \to 0 \). One can prove this in the following way:

\[
\lim_{h \to 0} (1 - h)^{t/h} = \exp \left[ \log \left( \lim_{h \to 0} (1 - h)^{t/h} \right) \right],
\]

(27)

\[
= \exp \left[ \lim_{h \to 0} \log(1 - h)^{t/h} \right],
\]

(28)

\[
= \exp \left[ \lim_{h \to 0} \log(1 - h) \right].
\]

(29)

One can use l’Hospital’s Rule, since both the numerator and the denominator in the equation (29) is zero when \( h = 0 \).

\[
\lim_{h \to 0} (1 - h)^{t/h} = \exp \left[ \lim_{h \to 0} \frac{\partial}{\partial h} \log(1 - h) \right],
\]

(30)

\[
= \exp \left[ \lim_{h \to 0} \frac{1 - h}{h} \right],
\]

(31)

\[
= \exp [-t].
\]

(32)

Hence the equation (26) becomes

\[
x(t) = \frac{x_0}{x_0 + (1 - x_0) e^{-t}}.
\]

(33)

in the limit where \( h \to 0 \). Equation (33) is same as the exact solution we obtained in section (b).
2 Limit Cycle

Equation (5, homework) can be expressed in polar coordinates by using the substitutions

\[ x = r \cos \theta, \quad (34) \]
\[ y = r \sin \theta, \quad (35) \]
\[ \dot{x} = \dot{r} \cos \theta - r \sin \theta \dot{\theta}, \quad (36) \]
\[ \dot{y} = \dot{r} \sin \theta + r \cos \theta \dot{\theta}. \quad (37) \]

Thus the equation (5, homework) in polar coordinates becomes

\[ \dot{r} \cos \theta - r \sin \theta \dot{\theta} = r \sin \theta + r \cos \theta f(r), \quad (38) \]
\[ \dot{r} \sin \theta + r \cos \theta \dot{\theta} = -r \cos \theta + r \sin \theta f(r). \quad (39) \]

By multiplying both sides of the equations (38) and (39) by \( \cos \theta \) and \( \sin \theta \), respectively, then adding them, one obtains

\[ \dot{r} = rf(r). \quad (40) \]

Similarly one can derive

\[ \dot{\theta} = -1. \quad (41) \]

Equation (41) means that the system \( \{x(t), y(t)\} \) rotates around origin with a constant angular velocity \(-1\). We can now find the solutions of equation (40) for three cases of \( f(r) \).

For \( f(r) = 1 - r^2 \), equation (40) becomes

\[ \frac{dr}{dt} = r(1 - r^2). \quad (42) \]

One can write this equation as

\[ dr \left( \frac{1}{r} + \frac{1}{2} \left( \frac{1}{1 - r} - \frac{1}{1 + r} \right) \right) = dt. \quad (43) \]

Carrying out the integral on both sides

\[ \int_{r_0}^{r(t)} dr \left( \frac{1}{r} + \frac{1}{2} \left( \frac{1}{1 - r} - \frac{1}{1 + r} \right) \right) = \int_0^t dt', \quad (44) \]
yields

\[
\log r + \frac{1}{2} \left( -\log(1 - r) - \log(1 + r) \right) \bigg|_{r_0}^{r(t)} = t, \\
\log r - \frac{1}{2} \log(1 - r^2) \bigg|_{r_0}^{r(t)} = t, \\
\log \frac{r}{\sqrt{1 - r^2}} \bigg|_{r_0}^{r(t)} = t, \\
\log \frac{r(t)}{\sqrt{1 - r^2(t)}} - \log \frac{r_0}{\sqrt{1 - r_0^2}} = t, \\
\frac{r(t)}{\sqrt{1 - r^2(t)}} = \frac{r_0}{\sqrt{1 - r_0^2}} e^t.
\]

This expression can be reorganized to get

\[
r(t) = \frac{r_0}{\sqrt{r_0^2 + (1 - r_0^2) e^{-2t}}}. \tag{50}
\]

From equation (50), one can see that \(r(t)\) converges to 1 regardless of the initial condition, except \(r_0 = 0\). Eventually the system begins to move on a circle of radius 1. On the other hand \(r = 0\) is an unstable stationary point: if the initial condition is \(r_0 = 0\), \(r(t)\) does not change in time, but for \(r_0 > 1\), it moves to 1.

For \(f(r) = r^2 - 1\), we get the same solution as above if we replace \(t\) with \(-t\). Therefore the solution is

\[
r(t) = \frac{r_0}{\sqrt{r_0^2 + (1 - r_0^2) e^{2t}}}. \tag{51}
\]

The behavior of this solution for different initial conditions can be summarized:

\[
\lim_{t \to \infty} r(t) = \begin{cases} 
\infty & r_0 > 1 \\
1 & r_0 = 1 \\
0 & r_0 < 1 
\end{cases} \tag{52}
\]

This tells us that the points on the circle of radius \(r = 1\) and whose center at the origin are unstable stationary points whereas the origin, \(r = 0\) is a stable stationary point.

In case of \(f(r) = (1 - r^2)^2\) the equation (40) reads

\[
\frac{dr}{dt} = r(1 - r^2)^2. \tag{53}
\]

Gathering each variable on the both sides separately and integrating yields

\[
\int_{r_0}^{r(t)} \frac{dr^2}{2r^2(1 - r^2)^2} = \int_0^t dt'. \tag{54}
\]
By using the substitutions

\[ r^2 = \frac{R}{(R - 1)}, \quad (55) \]

\[ dr^2 = -\frac{dR}{(1 - R)^2}, \quad (56) \]

one can get

\[ \int_{R_0}^{R(t)} \left( \frac{1}{R} - 1 \right) dR = 2t, \quad (57) \]

\[ \log R - R|_{R_0}^{R(t)} = 2t, \quad (58) \]

\[ \log \frac{R}{e^{R(t)}} \bigg|_{R_0}^{R(t)} = 2t, \quad (59) \]

\[ \frac{R(t)}{e^{R(t)}} = \frac{R_0}{e^{R_0}} e^{2t}. \quad (60) \]

Let us call this function \( G(r) \), then the solution is

\[ G(R(t)) = \frac{R(t)}{e^{R(t)}} = \frac{R_0}{e^{R_0}} e^{2t} \quad (61) \]

where \( R(t) = r(t)^2/(r^2(t) - 1) \). From equation (61), one can see that \( G(R) \) has always the tendency to increase in time for positive initial values \( G(R_0) > 0 \) or to decrease for negative values of \( G(R_0) \).

From equation (53), we know that the stationary points are \( r = 0 \) and \( r = 1 \). If \( r_0 = 0 \), then \( R_0 = G(R_0) = 0 \) and the system does not move from \( r = 0 \). Where \( 1 > r_0 > 0 \), \( G(R_0) \) is negative and the system moves towards \( r = 1 \), because \( \lim_{r(t)\to 1^-} G(R(t)) = -\infty \). So \( r = 0 \) is an unstable stationary point whereas \( r = 1^- \) is a stable stationary point (\( 1^- \) corresponds to a value that approaches 1 from left). The behavior of the system is depicted in figure 1, based on the function \( G(r) \).

If the system is at a point such that \( r > 1 \), which means \( G(R) > 0 \) then the system goes away from the point \( r = 1 \), because \( r \) has to increase in order for \( G(R) \) to increase by time, which is the condition imposed by equation (61). So the \( r = 1^+ \) is an unstable stationary point.

3 Bonhoefer-van der Pol Equation

(a)

Since \( \dot{x}_1(t) \) and \( \dot{x}_2(t) \) are zero at the stationary points \( (x_{1s}, x_{2s}) \), equation (6, homework) becomes

\[ f_1(x_{1s}, x_{2s}) = c \left( x_{2s} + x_{1s} - \frac{x_{1s}^3}{3} + z \right) = 0, \quad (62) \]

\[ f_2(x_{1s}, x_{2s}) = -\frac{1}{c} \left( x_{1s} + bx_{2s} - a \right) = 0. \quad (63) \]
Figure 1: $G(r) \text{ vs. } r$

From these equations we get

$$x_{2s} = \frac{1}{b}(a - x_{1s}) \quad (64)$$

$$0 = \frac{1}{b}(a - x_{1s}) + x_{1s} - \frac{x_{1s}^3}{3} + z \quad (65)$$

which has one real solution $(x_{1s}, x_{2s})$ for a certain value of $z$. It is plotted in figure 2 for values of $z$ between $-2$ and $2$.

Figure 2: Stationary points $x_{1s}$ and $x_{2s}$ vs. $z$

Elements of the matrix $M$ in the equation $(\delta x_1 = x_1 - x_{1s}, \delta x_2 = x_2 - x_{2s})$,

$$\delta \dot{x} = M\delta x, \quad (66)$$
can be calculated by using \( M_{jk} = \partial_k f_j(x_1, x_2)|_{x_s} \),
\[
M = \begin{pmatrix} c(1 - x_1^2) & c \\ -1/c & -b/c \end{pmatrix}.
\] (67)

If the eigenvalues and the eigenvectors of the matrix \( M \) are denoted by \( \lambda_{1,2} \) and \( m_{1,2} \), respectively, then the solution for the equation (66) is given by
\[
\delta x(t) = c_1 m_1 e^{\lambda_1 t} + c_2 m_2 e^{\lambda_2 t}.
\] (68)

where \( c_1 \) and \( c_2 \) are determined by the initial condition \( \delta x(0) \). One can easily tell from this equation that around a stable stationary point \((x_{1s}, x_{2s})\), \( \lambda_{1,2} \) must have negative real parts.

For \( z = 0 \), the stationary point is \( x_s = (1.20, -0.62) \). The corresponding matrix and the eigenvalues are given by
\[
M = \begin{pmatrix} -1.32 & 3 \\ -0.33 & -0.27 \end{pmatrix},
\] (69)
\[
\lambda_{1,2} = -0.79 \pm 0.85 i.
\] (70)

So the stationary point \( x_s = (1.20, -0.62) \) is stable due to the \( e^{-0.79t} \) term in the solution. With similar arguments, for \( z = -0.4 \), the stationary point \( x_s = (0.91, -0.26) \) is found to be unstable, because the real part of the eigenvalues of the corresponding matrix \( M \),
\[
M = \begin{pmatrix} 0.53 & 3 \\ -0.33 & -0.27 \end{pmatrix},
\] (71)
are positive, i.e.,
\[
\lambda_{1,2} = 0.13 \pm 0.92 i.
\] (72)

(c)

Using the discretized form of the differential equation given in the problem,
\[
x_{n+1} = x_n + \Delta t f(x_n),
\] (73)
one can get curves like in figure 3 for \( z = 0 \) and \( z = -0.4 \), with several starting points. Corresponding stationary points for these \( z \) values, i.e., \((1.20, -0.62)\) and \((0.91, -0.26)\) are marked in the plots.

(d)

When we include the noise term,
\[
x_{n+1} = x_n + \Delta t f(x_n) + \sigma \sqrt{\Delta t} \begin{pmatrix} \zeta_1(n) \\ \zeta_2(n) \end{pmatrix}
\] (74)
trajectories will look like in figure 4. In mathematica, one way of getting gaussian distributed random numbers, \( \zeta_j(n) \), is using the following code:
```
<< Statistics'ContinuousDistributions'
ndist = NormalDistribution[0, 1] %
\( \zeta = \text{Random}[\text{ndist}] \)
```
x_1(t) for the case \( z = -0.4 \) is plotted separately in figure 5.
Figure 3: $x_1(t), x_2(t)$ trajectories for different starting points (open ends of the trajectories). In case of a stable stationary point, all the trajectories come to rest at the stationary point whereas in case of an unstable stationary point the system begins a cyclic motion, regardless of its starting point.

Figure 4: $x_1(t), x_2(t)$ trajectories of the system, in case of gaussian random noise for several initial conditions ($\sigma = 0.15$, $\Delta t = 0.01$).

4 Cable Equation

Solution for the cable equation

$$v(x, t) = \sqrt{\frac{2}{a}} \sum_{n=1,3,\ldots} \alpha_n(t) \cos \frac{n\pi x}{2a}$$

obeys the boundary conditions, i.e.,

$$\left. \partial_x v(x, t) \right|_{x=0} = \sqrt{\frac{2}{a}} \sum_{n=1,3,\ldots} \frac{n\pi}{2a} \alpha_n(t) \sin \frac{n\pi x}{2a} \bigg|_{x=0},$$

$$= 0 \quad \text{for } n = 1, 3, \ldots$$
Figure 5: $x_1(t)$ vs. $t$, for the case $z = -0.4$. ($\sigma = 0.15$, $\Delta t = 0.01$).

and

$$v(a, t) = \sqrt{\frac{2}{a}} \sum_{n=1,3,...} \alpha_n(t) \cos \frac{n\pi}{2a},$$  \hspace{1cm} (78)

$$= 0.$$  \hspace{1cm} (79)

Inserting the general solution (75) into the cable equation provides us with the solution that governs $\alpha_n(t)$:

$$0 = (\partial_t - \partial_x^2 + 1)v(x, t) \hspace{1cm} (80)$$

$$= \sqrt{\frac{2}{a}} \sum_{n=1,3,...} \left( \dot{\alpha}_n(t) \cos \frac{n\pi x}{2a} + \alpha_n(t) \left( \frac{n\pi}{2a} \right)^2 \cos \frac{n\pi x}{2a} + \alpha_n(t) \cos \frac{n\pi x}{2a} \right) \hspace{1cm} (81)$$

$$= \sqrt{\frac{2}{a}} \sum_{n=1,3,...} \left( \dot{\alpha}_n(t) + \left( \frac{n\pi}{2a} \right)^2 \alpha_n(t) + \alpha_n(t) \right) \cos \frac{n\pi x}{2a} \hspace{1cm} (82)$$

In order for above summation to vanish, the term in the parenthesis must vanish. One can prove this by integrating both sides of the equation (82) with $\cos(\frac{m\pi x}{2a})$ where $m = 1, 3, 5,...$:

$$0 = \sqrt{\frac{2}{a}} \sum_{n=1,3,...} b_n \int_0^a \cos \frac{n\pi x}{2a} \cos \frac{m\pi x}{2a} dx \hspace{1cm} (83)$$

where $b_n$ is the term in the parenthesis in equation (82). The integral in the equation (83) can be calculated by using the identity

$$\cos A \cos B = \frac{1}{2} \left( \cos(A + B) + \cos(A - B) \right).$$

$$\int_0^a dx \cos \frac{n\pi x}{2a} \cos \frac{m\pi x}{2a} = \frac{1}{2} \left( \int_0^a dx \cos \frac{(m + n)\pi x}{2a} + \int_0^a dx \cos \frac{(m - n)\pi x}{2a} \right) \hspace{1cm} (84)$$

$$= \frac{a}{2} \left( \frac{\sin \frac{(m+n)\pi}{2}}{(m+n)\pi} + \frac{\sin \frac{(m-n)\pi}{2}}{(m-n)\pi} \right) \hspace{1cm} (85)$$
Let us define \( m = n + k \) where \( k \) can be any even integer, since \( m \) and \( n \) are both odd. With this substitution, the equation (85) becomes

\[
\int_0^a dx \cos \frac{n \pi x}{2a} \cos \frac{m \pi x}{2a} = \frac{a}{2} \left( \frac{\sin(n + \frac{k}{2}) \pi}{(n + \frac{k}{2}) \pi} + \frac{\sin \frac{k \pi}{2}}{\frac{k \pi}{2}} \right) \quad (86)
\]

\[
= \frac{a}{2} \left( -\frac{\sin \frac{k \pi}{2}}{(n + \frac{k}{2}) \pi} + \frac{\sin \frac{k \pi}{2}}{\frac{k \pi}{2}} \right) \quad (87)
\]

\[
= \frac{a}{2} \left( \frac{\sin \frac{k \pi}{2}}{\frac{k \pi}{2}} \right) \quad (88)
\]

\[
= \begin{cases} 
\frac{a}{2} & k = 0 \ (i.e., m = n) \\
0 & k \neq 0 \ (i.e., m \neq n) 
\end{cases} \quad (89)
\]

In other words, one can state

\[
\int_0^a dx \cos \frac{n \pi x}{2a} \cos \frac{m \pi x}{2a} = \frac{a}{2} \delta_{mn}. \quad (90)
\]

From equations (eqn.need.proof.proved) and (83), one gets

\[
0 = \sqrt{\frac{a}{2}} \sum_{n=1,3,...} b_n \delta_{mn}, \quad (91)
\]

from which we obtain

\[
b_m = 0 \quad (92)
\]

where \( m = 1, 3, 5, ... \). Thus we have proved that each and every term in the parenthesis of the summation in the equation (82) must be zero, i.e.,

\[
\dot{\alpha}_n(t) = - \left( 1 + \frac{n^2 \pi^2}{4a^2} \right) \alpha_n(t). \quad (93)
\]

Solution for this equation is given by

\[
\alpha_n(t) = \alpha_n(0) \exp \left[ - \left( 1 + \frac{n^2 \pi^2}{4a^2} \right) t \right]. \quad (94)
\]

The coefficients, \( \alpha_n(0) \), are determined through the initial condition,

\[
v(x,0) = \sum_{n=1,3,...} \alpha_n(0) \sqrt{\frac{2}{a}} \cos \frac{n \pi x}{2a}. \quad (95)
\]

By integrating both sides with \( \cos(m \pi x/2a) \), one gets

\[
\int_0^a v(x,0) \cos \frac{m \pi x}{2a} dx = \sum_{n=1,3,...} \sqrt{\frac{2}{a}} \alpha_n(0) \int_0^a \cos \frac{m \pi x}{2a} \cos \frac{n \pi x}{2a} dx. \quad (96)
\]
From the equations (90) and (96), we obtain

$$1 = \sqrt{\frac{a}{2}} \sum_{n=1,3,...} \alpha_n(0) \delta_{nm}. \quad (97)$$

We can now write the coefficient $\alpha_m(0)$ as

$$\alpha_m(0) = \sqrt{\frac{2}{a}}. \quad (98)$$

Consequently using equations (98), (94) and (75), one obtains the complete solution,

$$v(x,t) = \frac{2}{a} \sum_{n=1,3,...} \cos \frac{n\pi x}{2a} \exp \left[-\left(1 + \frac{n^2\pi^2}{4a^2}\right)t\right]. \quad (99)$$

$v(x,t)$ is plotted in figure 6.

Figure 6: $v_{1,2,3,4}(x,t)$ vs. $x,t$. Higher modes, in other words, narrower cosines, die off quickly by the time, so the initial Dirac delta function broadens while its magnitude decreases (see equation (99)).