Problem Set 2
Physics498: Theoretical Biophysics /Spring 2002
Professor Klaus Schulten

Problem 1: Verhulst Equation

Consider the evolution equation

$$\dot{x} = x - x^2 \tag{1}$$

that describes the potential $V(t) = x$ of a neuron.

(a) Determine the stationary points x_{1s}, x_{2s} and the behaviour of the solution in linear approximation $x_{1s},\ x_{2s} = a,\ \delta x = x - a,\ f(x) = x - x^2$

$$\delta\dot{x} = f(a) + \partial_x f|_a \delta x \tag{2}$$

(b) Derive the exact solution of eq.(1) for arbitrary initial condition $x(0) = x_0$. Compare the result with the approximation considered in (a).
(c) For a numerial solution of eq. (1) one discretizes the time-derivative in eq.(1) through $(x_{n+1} - x_n)/\Delta t$ for $t \to t_n = n\Delta t$ and $x_n = x(t_n)$. Assume that the differential equation (1) is replaced by

$$x_{n+1} - x_n = h(x_{n+1} - x_n x_{n+1}) \tag{3}$$

Show that the solution of this equation is

$$x_n^{-1} = 1 - (1 - x_0^{-1})(1 - h)^n \tag{4}$$

Discuss the behaviour of this solution for $h > 0,\ h \to 0$ and for $h < 1$ and $h \geq 1$.

Problem 2: Limit Cycle

Discuss the solution of the equation

$$\begin{aligned}\dot{x} &= y + xf(r)\\ \dot{y} &= -x + yf(r)\end{aligned} \tag{5}$$

($r = \sqrt{x^2+y^2}$) for the functions $f(r) = (1-r^2)$, (r^2-1) and $(r^2-1)^2$. (Hint: Use polar coordinates).

Problem 3: Bonhoefer-van der Pol Equation

Consider the solution of the evaluation equation ($c = 3,\ a = 0.7,\ b = 0.8$)

$$\begin{aligned}\dot{x}_1 &= c(x_2 + x_1 - \frac{x_1^3}{3} + z) = f_1(x_1, x_2)\\ \dot{x}_2 &= -\frac{1}{c}(x_1 + bx_2 - a) = f_2(x_1, x_2)\end{aligned} \tag{6}$$

(a) Discuss the stationary point $\mathbf{x}_s$ of this system of equations in the range $-2 \leq z \leq 2$, i.e $\mathbf{f}(\mathbf{x}_s) = 0$.

(b) For $z = 0$ and $z = -0.4$ carry out a linear stability analysis, i.e. solve the equation ($\delta\mathbf{x} = \mathbf{x} - \mathbf{x}_s$)

$$\delta\dot{\mathbf{x}} = \mathbf{M}\delta\mathbf{x} \tag{7}$$

for $M_{jk} = \partial_k f_j(\mathbf{x}_s)$. For this purpose expand the solutions in terms of the eigenvectors of $\mathbf{M}$.
Note: The eigenvectors for different eigenvalues may not be orthogonal!

(c) Determine typical trajectories $\mathbf{x}(t)$ for $z = 0$ and $z = -0.4$ for a variety of initial points $\mathbf{x}(0)$. For this purpose discretize the system eqs. (6) in the form

$$\mathbf{x}_{n+1} = \mathbf{x}_n + \Delta t f(\mathbf{x}_n) \tag{8}$$

for suitable Δt. Plot the result.

(d) Consider now eq. (8) in the form

$$\mathbf{x}_{n+1} = \mathbf{x}_n + \Delta t f(\mathbf{x}_n) + \sigma\sqrt{\Delta t}\begin{pmatrix}\zeta_1(n)\\ \zeta_2(n)\end{pmatrix} \tag{9}$$

for $\sigma = 0.15$ and for random numbers $\zeta_1(n),\ \zeta_2(n)$ that are Gaussian distributed, i.e. $p(\zeta_i) = 1/\sqrt{2\pi}\exp[-\zeta_i^2]$. Plot the results.

Problem 4: Cable Equation

Solve the cable equation

$$(\partial_t - \partial_x^2 + 1)v(x,t) = 0 \tag{10}$$

for $x \in [0, a]$, the boundary conditions

$$\begin{aligned} \partial_x v &= 0 \;, \quad x = 0 & (11) \\ v(a) &= 0 & (12) \end{aligned}$$

and the initial condition

$$v(x, 0) = \delta(x) \;\; . \tag{13}$$

For this purpose expand the solutions in terms of eigenfunctions of ∂_x^2 that obey the boundary conditions, i.e.

$$f_n(x) = \sqrt{\frac{2}{a}} \cos \frac{\pi n x}{2a} \tag{14}$$

$n = 1, 3, 5, ...$, i.e. set $v(x,t) = \sum_{n=1,3,5,...} \alpha_n(t) f_n(x)$. Determine the $\alpha_n(t)$ that obey the initial condition eq. (13). Plot the result.

Due Thursday April 18th in class