Project: $U(3)$ and the Oscillator
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Problem: The $U(3)$ symmetry of the isotropic harmonic oscillator

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This problem studies the $U(3)$ symmetry of the three dimensional isotropic harmonic oscillator. The $3D$ isotropic harmonic oscillator is described by the following Hamiltonian

$$\begin{aligned} H &= \sum_{k=1}^{3}\left(\frac{p_k^2}{2m} + \frac{1}{2}m\omega^2 x_k^2\right) \\ &= \hbar\omega \sum_{k=1}^{3}\left(a_k^\dagger a_k + \frac{1}{2}\right) \end{aligned} \tag{1}$$

where $a_k^\dagger$ and a_k are the creation and annihilation operators defined as

$$a_k = \sqrt{\frac{m\omega}{2\hbar}}x_k + i\frac{1}{\sqrt{2m\hbar\omega}}p_k \tag{2}$$

which satisfy the commutation relations

$$\begin{aligned} [a_j, a_k^\dagger] &= \delta_{jk}, \\ [a_j, a_k] &= 0, \\ [a_j^\dagger, a_k^\dagger] &= 0. \end{aligned} \tag{3}$$

As an early warm up we want to compute the energy levels *and* the degeneracies for this Hamiltonian from the knowledge of the single dimensional harmonic oscillator.

(a) Show that the energy level $E_n = \hbar\omega(n + \frac{3}{2})$ is $(n+1)(n+2)/2$ times degenerate. (*Hint*: Use induction on the dimension of the oscillator.)

In this exercise we will study the $U(3)$ symmetry of the isotropic harmonic oscillator.

(b) Show that the Hamiltonian is invariant under transformations of the form

$$a_k \to U_{kl}a_l \tag{4}$$

for a 3×3 unitary matrix U.

In order ro study the $U(3)$ symmetry of the harmonic oscillator we want to define the so-called *shift operators*

$$G_{ij} = \frac{1}{2}(a_i^\dagger a_j + a_j a_i^\dagger). \tag{5}$$

(c) Show that the shift operators satisfy

$$[G_{ij}, G_{kl}] = \delta_{jk} G_{il} - \delta_{il} G_{kj}. \tag{6}$$

(d) Show that the Hamiltonian can be written as

$$H = \hbar\omega(G_{11} + G_{22} + G_{33}), \tag{7}$$

and that it commutes with all of the G_{jk}.

Now let us introduce a new basis of generators, $\tilde{\lambda}_k$, as follows

$$\begin{aligned}
\tilde{\lambda}_1 &= G_{12} + G_{21} && (8)\\
\tilde{\lambda}_2 &= -iG_{12} + iG_{21} && (9)\\
\tilde{\lambda}_3 &= G_{11} - G_{22} && (10)\\
\tilde{\lambda}_4 &= G_{13} + G_{31} && (11)\\
\tilde{\lambda}_5 &= -iG_{13} + iG_{31} && (12)\\
\tilde{\lambda}_6 &= G_{23} + G_{32} && (13)\\
\tilde{\lambda}_7 &= -iG_{23} + iG_{32} && (14)\\
\tilde{\lambda}_8 &= \frac{1}{\sqrt{3}}(G_{11} + G_{22} - 2G_{33}). && (15)
\end{aligned}$$

(e) Show that the set of generators, $\tilde{\lambda}_k$ form the same algebra as the λ_k operators of the standard $SU(3)$ algebra, introduced in section 12.3. Namely, show that

$$[\tilde{\lambda}_j, \tilde{\lambda}_k] = 2i f_{jkl} \tilde{\lambda}_l, \tag{16}$$

where f_{jkl} are the same structure constants as in the $SU(3)$ algebra. (*Hint*: There is a less painful way to show this without computing a single f_{jkl}. Let M_{jk} be the matrix all whose elements are zero except for the $j-k$ element, which is unity. (i.e. $[M_{jk}]_{mn} = \delta_{jm}\delta_{kn}$.) Then one can show that M_{jk} and

G_{jk} satisfy the same algebra and that λ_k are just matrix representations of $\tilde{\lambda}_k$.)

[The interested reader is invited to experiment with two mathematica notebooks that contain an explicit framework to handle arbitrary Lie algebras. (They are *not* necessary for this exercise.) In particular $SU(3)$ has been studied there in painstaking detail. They are to be found at:
http://www.ks.uiuc.edu/Services/Class/PHYS481/u3_algebra.nb and
http://www.ks.uiuc.edu/Services/Class/PHYS481/u3_algebra_II.nb]

The group $U(3)$ has 9 generators as embodied by G_{ij}. We know that one linear combination of them is the Hamiltonian (the $U(1)$ part of $U(3)$) and three more are the generators of the rotation group (as the problem is spherically symmetric). A natural question would be to ask what the physical meaning of the other extra symmetry generators are. (e.g. In the case of the hydrogen atom, studied in the beginning of the chapter, the extra symmetry generators were the components of the eccentricity vector.) In order to answer this question we first need to write the generators G_{ij} in terms of familiar physical operators:

(f) Show that the shift operators can be expressed as

$$G_{ij} = \frac{m\omega}{2\hbar} x_i x_j + \frac{1}{2m\hbar\omega} p_i p_j - \frac{1}{2}\epsilon_{ijk} L_k, \tag{17}$$

where $L_k = \frac{1}{i\hbar} J_k$.

Now, we want to express the cartesian tensor, G_{ij}, in the form of a spherical tensor. (Please refer to section 6.7 for the subject.)
The *spherical decomposition* of G_{ij} is defined by

$$G_{ij} = \frac{G_{kk}}{3}\delta_{ij} + \frac{G_{ij} - G_{ji}}{2} + \left(\frac{G_{ij} + G_{ji}}{2} - \frac{G_{kk}}{3}\delta_{ij}\right) \tag{18}$$

In other words, we write a cartesian tensor, respectively, as a trace-only part, an anti-symmetric part and a traceless symmetric part. (Note that dimensions add up nicely: $3 \times 3 = 1 + 3 + 5$.)

(g) Show that the first two terms in (18) correspond, respectively, to the Hamiltonian (times a prefactor), $\frac{1}{3\hbar\omega}H$, and the angular momentum operator, L_k. (We will shortly prove that the last term is the quadrupole moment operator, Q_m.)

To study the last term in (18) we define

$$\frac{1}{3}Q_{ij} \equiv \frac{G_{ij}+G_{ji}}{2} - \frac{G_{kk}}{3}\delta_{ij}, \tag{19}$$

which is a traceless and symmetric tensor, which therefore has only 5 degrees of freedom. We want to rewrite the components of Q_{ij} as a spherical tensor:

$$Q_0 = Q_{zz}, \tag{20}$$

$$Q_{\pm 1} = -\sqrt{\frac{2}{3}}\left(iQ_{yz} \pm Q_{xz}\right), \tag{21}$$

$$Q_{\pm 2} = \sqrt{\frac{1}{6}}(Q_{xx}-Q_{yy}) \pm i\sqrt{\frac{2}{3}}Q_{xy}. \tag{22}$$

The choice of prefactors are not arbitrary: they are choosen in such a way to ensure that Q_m transform like the spherical harmonics.

(h) Show that the quadrupole moment operator, Q_m, is a rank 2 spherical tensor whose components are given by

$$Q_m = \sqrt{\frac{16\pi}{5}}\left(\frac{m\omega}{2\hbar}Y^2_m(\widehat{x})x^2 + \frac{1}{2m\hbar\omega}Y^2_m(\widehat{p})p^2\right), \quad m = \pm 2, \pm 1, 0. \tag{23}$$

For your convenience the necessary spherical harmonics are given below:

$$Y^2_0(\hat{x}) = \sqrt{\frac{5}{16\pi}}\frac{3z^2-r^2}{r^2}, \tag{24}$$

$$Y^2_{\pm 1}(\hat{x}) = \mp\sqrt{\frac{15}{8\pi}}\frac{(x \pm iy)z}{r^2}, \tag{25}$$

$$Y^2_{\pm 2}(\hat{x}) = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\frac{(x \pm iy)^2}{r^2}. \tag{26}$$

Finally we will try to understand the degeneracy structure of the isotropic three dimensional harmonic oscillator starting from its symmetries. In order to accomplish this we will need the following Casimir operator

$$C = \sum_{ij} G_{ij}G_{ji}, \tag{27}$$

$$= -\frac{1}{2}L^2 + \frac{1}{6}Q^2. \tag{28}$$

To obtain a more explicit expression one may use the following identity

$$Q^2 = Q_0^2 - Q_{+1}Q_{-1} - Q_{-1}Q_{+1} + Q_{+2}Q_{-2} + Q_{-2}Q_{+2}. \tag{29}$$

The Casimir operator introduced above can be written in terms of the Hamiltonian as follows:

$$C = -3 + \frac{4}{3\hbar^2\omega^2} H^2. \tag{30}$$

(This can be proven after a some algebra but you don't have to do this.)

(i)For a given energy eigenstate with energy $\hbar\omega(n + \frac{3}{2})$ show that

$$C = \frac{4}{3}(n^2 + 3n). \tag{31}$$

We will state without proof that the value of the Casimir operator on any given irreducible representation, $D(\mu, \nu)$, of the $SU(3)$ group is given by[1]

$$C = \frac{4}{3}(\mu^2 + \mu\nu + \nu^2 + 3\mu + 3\nu). \tag{32}$$

Comparing (31) with (32) we see that only the representations with $(\mu, \nu) = (n, 0)$ are realized by the oscillator. Hence the degeneracy of the energy levels for a given energy level, E_n, should be equal to the dimension of the representation $D(n, 0)$.

(j) Using the formula for the dimension of a representation $D(\mu = n, \nu = 0)$ from section 12.3, derive the degeneracy for the energy level $\hbar\omega(n + \frac{3}{2})$. Compare this with your result from the beginning of the exercise.

[1]A proof of this relation may be found in *Microscopic theory of the nucleus*, J. M. Eisenberg and W. Greiner, 1972, North-Holland, section 8.2. The derivation, which uses Young tableaux and a fair amount of representation theory, is too long to be included here.