Chapter 12

Symmetries in Physics: Isospin and the Eightfold Way

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Symmetries and their consequences are central to physics. In this chapter we will discuss a particular set of symmetries that have played a seminal role in the development of elementary particle and nuclear physics. These are the *isospin symmetry* of nuclear interactions and its natural extension, the so-called *Eightfold Way*.
The organization of this chapter is as follows: In the next section we will discuss the relation between symmetries of a quantum mechanical system and the degeneracies between its energy levels. We will particularly use the example of spherically symmetric potentials. In the following section we will introduce the concept of isospin as an approximate $SU(2)$ symmetry, which identifies the proton and the neutron as different states of the same particle. We will also introduce the quark model as a natural framework to represent the observed symmetries. We will apply these concepts to an analysis of nucleon-nucleon and nucleon-meson scattering. In the final section, we will discuss the $SU(3)$ symmetry of three quark flavors. The algebraic structure and the representations of $SU(3)$ will be discussed in parallel to $SU(2)$ and particle families will be identified in terms of representations of the underlying symmetry group.

12.1 Symmetry and Degeneracies

The degeneracies of energy levels of a quantum mechanical system are related to its symmetries. Let us assume a continuous symmetry obeyed by a quantum mechanical system. The action of the symmetry operations on quantum mechanical states are given by elements of a corresponding Lie group, i.e.,

$$\mathcal{O} = \exp\left(\sum_k \alpha_k S_k\right). \tag{12.1}$$

Then the generators, S_k, will commute with the Hamiltonian of the system,

$$[H, S_k] = 0. \tag{12.2}$$

The action of any symmetry generator, S_k, on an energy eigenstate, $\psi_{E,\lambda_1,\dots,\lambda_n}$, leaves the energy of the state invariant

$$H \exp(i\alpha_k S_k)\, \psi_{E,\lambda_1,\dots,\lambda_n} \;=\; E \exp(i\alpha_k S_k)\, \psi_{E,\lambda_1,\dots,\lambda_n}. \tag{12.3}$$

If the newly obtained state is linearly independent of the original one, this implies a degeneracy in the spectrum. We will investigate this shortly in more detail in the case of systems with spherical symmetry, where the symmetry generators, S_k, can be identified with the angular momentum operators, J_k, as studied in chapter 7.
Lie groups play an essential role in the discussion of mass degeneracies in particle physics. In order to illustrate this, we first consider a particular example of the implications of symmetry, namely motion in a spherically symmetric potential described by the group $SO(3)$ (or its double covering $SU(2)$ as discussed in section 5.12).
In chapter 7 the dynamics of a particle moving in three dimensions under the influence of a spherically symmetric potential, $V(r)$, has been discussed. The spherical symmetry implies the commutation of the Hamiltonian with angular momentum operators (7.8)

$$[\hat{H},\, J_k] \;=\; 0\,, \quad k \;=\; 1,2,3\,. \tag{12.4}$$

The stationary Schrödinger equation, (7.18), can then be reduced to a one-dimensional radial equation, (7.24), which yields a set of eigenstates of the form

$$\psi_{E,\ell,m}(\vec{r}) \;=\; v_{E,\ell,m}(r)\, Y_{\ell m}(\theta,\phi), \tag{12.5}$$

with $m = -l, \cdots, l$ and the corresponding energy levels are independent of m. Therefore, each energy level is $(2l+1)$-fold degenerate. This degeneracy follows from the fact that any rotation, as represented by an element of $SO(3)$, (7.39), generates a state which has the same energy as the original one.

Presence of additional symmetries may further increase the degeneracy of the system. As an example of this we will consider the Coulomb problem with the Hamiltonian

$$H \;=\; \frac{\vec{p}^2}{2m} - \frac{k}{r}. \tag{12.6}$$

From elementary quantum mechanics we know the spectrum of the hydrogen atom. The energy levels are

$$E_n = -\frac{mk^2}{2\hbar^2 n^2}, \tag{12.7}$$

where the orbital angular momentum, l, is allowed to take values in $0,\dots,n-1$. The energy levels are totally independent of l. For example, the states $3s$, $3p$ and $3d$ all have the same energy. We want to understand this extra degeneracy in terms of the extra symmetry of the hydrogen atom given by an additional set of symmetry generators introduced below.
Classically, the additional symmetry generators of the Coulomb problem are the three components of the so-called *eccentricity* vector discovered by Hamilton

$$\vec{\epsilon} = \frac{1}{m}\vec{p}\times\vec{J} - k\frac{\vec{r}}{r}. \tag{12.8}$$

The vector $\vec{\epsilon}$ points along the symmetry axis of the elliptical orbit and its length equals the eccentricity of the orbit. The vector in (12.8) is not a hermitian operator. The corresponding quantum mechanical hermitian operator can be defined by

$$\epsilon_1 = \frac{1}{2m}(p_2 J_3 - p_3 J_2 + J_3 p_2 - J_2 p_3) - k\frac{x_1}{r}, \tag{12.9}$$

$$\epsilon_1 = \frac{1}{2m}(J_3 p_2 - J_2 p_3 + i\hbar p_1) - k\frac{x_1}{r}, \tag{12.10}$$

$$\epsilon_2 = \frac{1}{2m}(J_1 p_3 - J_3 p_1 + i\hbar p_2) - k\frac{x_2}{r}, \tag{12.11}$$

$$\epsilon_3 = \frac{1}{2m}(J_2 p_1 - J_1 p_2 + i\hbar p_3) - k\frac{x_3}{r}, \tag{12.12}$$

$$\tag{12.13}$$

It can be verified explicitly that its components commute with the Hamiltonian.
In order to understand the aforementioned extra degeneracy, we will compute the hydrogen spectrum using the additional symmetry. For this purpose we first note that

$$\vec{J}\cdot\vec{\epsilon} = 0. \tag{12.14}$$

This follows from $\vec{a}\cdot(\vec{a}\times\vec{b}) = (\vec{a}\times\vec{b})\cdot\vec{a} = 0$, which is valid even when $\vec{a}$ and $\vec{b}$ do not commute. We will also need the following identity [4]

$$\vec{\epsilon}^2 = \frac{2H}{m}\left(\vec{J}^2 + \hbar^2\right) + k^2, \tag{12.15}$$

which can be proved after very considerable algebra.
In the following we consider the bound states, which have a negative energy E. Therefore, in the subspace of the Hilbert space corresponding to a certain energy we can replace H by E. Now we scale the eccentricity vector as follows

$$\vec{K} = \sqrt{-\frac{m}{2E}}\vec{\epsilon}. \tag{12.16}$$

Through some algebra [4] the following commutation relations can be verified

$$[K_i, J_j] = i\hbar\epsilon_{ijk}K_k, \tag{12.17}$$

$$[K_i, K_j] = i\hbar\epsilon_{ijk}J_k, \tag{12.18}$$

which complement the familiar angular momentum algebra of section 5.3.
We introduce the following new operators

$$\vec{A} = \frac{1}{2}(\vec{J} + \vec{K}), \tag{12.19}$$

$$\vec{B} = \frac{1}{2}(\vec{J} - \vec{K}), \tag{12.20}$$

which can be shown to satisfy

$$[A_i, A_j] = i\hbar\epsilon_{ijk}A_k, \tag{12.21}$$

$$\begin{aligned} [B_i, B_j] &= i\hbar\epsilon_{ijk}B_k, &(12.22)\\ [A_i, B_j] &= 0, &(12.23)\\ \left[\vec{A}, H\right] &= 0, &(12.24)\\ \left[\vec{B}, H\right] &= 0. &(12.25) \end{aligned}$$

So far we have shown that the symmetry generators form an algebra, which is identical to the the direct sum of the Lie algebra of of two $SO(3)$ (or $SU(2)$) algebras. By comparing to the rotation algebra introduced in chapter 5, we can read off the eigenvalues of $\vec{A}^2$ and $\vec{B}^2$ from (12.21) and (12.22):

$$\begin{aligned} \vec{A}^2 &= a(a+1)\hbar^2\ ,\ a = 0, \frac{1}{2}, 1, \dots\ , &(12.26)\\ \vec{B}^2 &= b(b+1)\hbar^2\ ,\ b = 0, \frac{1}{2}, 1, \dots\ . &(12.27) \end{aligned}$$

Following (12.14) we note that

$$\vec{A}^2 - \vec{B}^2 = \vec{J}\cdot\vec{\epsilon} = 0. \tag{12.28}$$

This implies that $a = b$. In order to arrive at the spectrum a final bit algebra is needed

$$\begin{aligned} \vec{A}^2 + \vec{B}^2 &= \vec{J}^2 + \vec{K}^2 &(12.29)\\ &= \vec{J}^2 - \frac{m}{2E}\vec{\epsilon}^2 &(12.30)\\ &= -\frac{mk^2}{4E} - \frac{1}{2}\hbar^2, &(12.31) \end{aligned}$$

where we have used (12.15). Using this equation the energy eigenvalues can be written in terms of the eigenvalues of $\vec{A}^2$ and $\vec{B}^2$ operators. Noticing that $\vec{A}^2$ and $\vec{B}^2$ have the same eigenvalues because of (12.28), the energy eigenvalues are found to be

$$E = -\frac{mk^2}{2\hbar^2(2a+1)^2},\ a = 0, \frac{1}{2}, 1, \dots\ . \tag{12.32}$$

A comparison with (12.7) tells us that $(2a+1) = n$. Furthermore the bound on the orbital angular momentum, l, can be seen to follow from the triangle inequality as applied to $\vec{J} = \vec{A} + \vec{B}$, namely that

$$\begin{aligned} \left|\vec{J}\right| &> \left|\vec{A}\right| - \left|\vec{B}\right| = 0 &(12.33)\\ \left|\vec{J}\right| &< \left|\vec{A}\right| + \left|\vec{B}\right| = 2\left|\vec{A}\right| &(12.34) \end{aligned}$$

It follows that l has to have values in $\{0 = |a-b|, 1, \dots, a+b = n-1\}$. This illustrates the effect of additional symmetries to the degeneracy structure of a quantum mechanical system.

In contrast to the discussion above about extra symmetries, a lack of symmetry implies a lack of degeneracy in the energy levels of a quantum mechanical system. The most extreme case of this is the quantum analogue of a classically chaotic system. Chaos is described classically as exponential

sensitivity to initial conditions, in the sense that nearby trajectories in the phase space diverge from each other over time. However, another manifestation of chaos is the lack of independent operators commuting with the Hamiltonian.
A typical example of this so called quantum chaos is the quantum billiard problem, which is a particle in box problem in two dimensions with a boundary which can be chosen arbitrarily. If the chosen boundary is 'irregular' in a suitably defined sense, the classical trajectories will diverge from each other after successive bounces from the boundary. For a more detailed discussion of quantum chaos in billiard systems we refer the reader to [7] and the references therein.
In the case of billiards and other examples of quantum chaos one common observation is the almost nonexistence of degeneracies and the fact that the energy levels are more evenly spaced. This is known as *level repulsion*.

In the next section we will proceed with the discussion of a symmetry, which was discovered by observing degeneracies in the particle spectrum.

12.2 Isospin and the $SU(2)$ flavor symmetry

The concept of *isospin* goes back to Heisenberg, who, after the discovery of the neutron in 1932, suggested that the proton and the neutron can be regarded as two states of a single particle. This was motivated by the observation that their masses are approximately equal: $m_p = 938.28MeV/c^2$, $m_p = 939.57MeV/c^2$. Following the mass-energy equivalence of special relativity

$$E \; = \; mc^2, \tag{12.35}$$

this mass equivalence can be viewed as an energy degeneracy of the underlying interactions.
This (approximate) degeneracy led into the idea of the existence of an (approximate) symmetry obeyed by the underlying nuclear interactions, namely, that the proton and the neutron behave identically under the so-called *strong interactions* and that their difference is solely in their charge content. (Strong interactions bind the atomic nucleus together.)
If the proton and the neutron are to be viewed as two linearly independent states of the same particle, it is natural to represent them in terms of a two component vector, analogous to the spin-up and spin-down states of a spin-$\frac{1}{2}$ system

$$p = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; n = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{12.36}$$

In analogy to the concept of spin regarding the rotations in 3-space as discussed in chapter 5, the isospin symmetry is also governed by an $SU(2)$ group 'rotating' components in (12.36) into each other in abstract isospin space. This enables us to utilize what we already know about the $SU(2)$ symmetry group from the study of angular momentum. For example, we will be able to use the familiar Clebsch-Gordan coefficients to combine the isospin of two particles the same way we added spin in chapter 6.
It is important to place the isospin concept in its proper historical context. Originally it was believed that isospin was an exact symmetry of strong interactions and that it was violated by electromagnetic and weak interactions. (Weak interactions are responsible, for example, for the

beta decay). The mass difference between the neutron and the proton could then be attributed to the charge content of the latter. If the mass difference (or the energy difference) were to be to be purely electrostatic in nature, the proton had to be heavier. However, the proton is the lighter of the two. If it were otherwise the proton would be unstable by decaying into the neutron, spelling disaster for the stability of matter.
Isospin symmetry is *not* an exact symmetry of strong interactions, albeit it is a good approximate one. Therefore it remains a useful concept. Further than that, as we shall see below, it can be seen as part of a larger (and more approximate) symmetry which is of great utility to classify observed particle families.

We can describe isospin multiplets the same way we have described the angular momentum and spin multiplets. Denoting the total isospin, I, and its third component, I_3, as good quantum numbers, we can re-write (12.36) as a multiplet with $I = \frac{1}{2}$

$$p = \left| I = \frac{1}{2}, I_3 = \frac{1}{2} \right\rangle, \; n = \left| I = \frac{1}{2}, I_3 = -\frac{1}{2} \right\rangle. \tag{12.37}$$

As an example of a multiplet with $I = 1$ we have the three pions or π-mesons

$$\pi^+ = |1, 1\rangle, \; \pi^0 = |1, 0\rangle, \; \pi^- = |1, -1\rangle, \tag{12.38}$$

which have all nearly identical masses. ($m_{\pi^\pm} = 139.6 MeV/c^2$, $m_{\pi^0} = 135.0 MeV/c^2$). Shortly we will see how to describe both the pion and nucleon states as composites of more fundamental $I = \frac{1}{2}$ states.
In the framework of the quark of model, the fundamental representation of the isospin symmetry corresponds to the doublet that contains the so-called *up* and *down* quarks

$$u = \left| \frac{1}{2}, \frac{1}{2} \right\rangle, \; d = \left| \frac{1}{2}, -\frac{1}{2} \right\rangle. \tag{12.39}$$

All other isomultiplets, including the proton and the neutron, are made up of these two quarks. They can be constructed with the same rules that have been used for angular momentum addition in chapter 6. For example, the three pions in (12.38) are $u\bar{d}$, $u\bar{u}$ and $d\bar{u}$ states, respectively. They form an isotriplet:

$$\pi^+ = |1, 1\rangle = \left| \frac{1}{2}, \frac{1}{2} \right\rangle_1 \left| \frac{1}{2}, \frac{1}{2} \right\rangle_2, \tag{12.40}$$

$$\pi^0 = |1, 0\rangle = \frac{1}{\sqrt{2}} \left(\left| \frac{1}{2}, \frac{1}{2} \right\rangle_1 \left| \frac{1}{2}, -\frac{1}{2} \right\rangle_2 + \left| \frac{1}{2}, \frac{1}{2} \right\rangle_1 \left| \frac{1}{2}, -\frac{1}{2} \right\rangle_2 \right), \tag{12.41}$$

$$\pi^- = |1, -1\rangle = \left| \frac{1}{2}, -\frac{1}{2} \right\rangle_1 \left| \frac{1}{2}, -\frac{1}{2} \right\rangle_2. \tag{12.42}$$

$$\tag{12.43}$$

Similarly, the proton and the neutron can be written as totally symmetric *uud* and *udd* states. For a precise description of the two nucleons as composite states, including the spin and color quantum numbers of their constituent quarks, we refer the reader to [1], sec. 2.11.

The mass of the up and down quarks are *not* identical but they are both of the order of a few MeV/c^2's which is minuscule compared to the typical energy scale of hadrons (i.e. strongly interacting particles) which is about a GeV/c^2. This is why isospin is such a good symmetry and why isomultiplets have nearly identical masses.

As it later turned out, the up and down quarks are not the only quark 'species' - or *flavors* as they are commonly called. In the late 1940's and early 1950's, a number *strange* particles have been found which presumably contained a third quark species: the strange quark. It shall be noted here that the quark model was not invented until 1960's, but at the time the empirical concepts like isospin and *strangeness* quantum numbers were in use. The value of the strangeness quantum number is taken, by accidental convention, to be -1 for the strange quark. The up and down quarks have strangeness zero. All other composite states have their strangeness given by the sum of the strangeness content of their constituents.

Before proceeding further, we shall setup some terminology: *baryons* are qqq states, such as proton and the neutron, whereas *mesons* are $q\bar{q}$ states, the pions being examples thereof. By convention baryons have *baryon number* 1, and quarks have baryon number $\frac{1}{3}$. All antiparticles have their quantum numbers reversed. Naturally, mesons have baryon number 0. The names, baryon and meson, originally refer to the relative weight of particles, baryons generally are heavy, mesons have intermediate mass ranges, where *leptons* (electron, muon, the neutrinos etc.) are light. If taken literally, this remains only an inaccurate naming convention today, as some mesons discovered later are heavier than some baryons and so on.

The relation between electric charge and isospin are given by the *Gell-Mann–Nishijima relation* which was first derived empirically

$$Q = I_3 + \frac{1}{2}(B + S), \tag{12.44}$$

where B is the baryon number and S is the strangeness. In the next section we will be able to view the Gell-Mann–Nishijima relation in the light of the representation theory for the flavor $SU(3)$ symmetry.

Now let us consider another example of combining the isospins of two particles. The reader may know that the *deuteron*, a hydrogen isotope, consists of a proton and a neutron. Therefore it has to have isospin, $I_3 = 0$. We will now try to describe its wave function in terms of its constituent nucleons. Following (12.37) and in analogy to (12.43), this will be mathematically identical to adding two spins. The possibilities are that of an *isosinglet*

$$|0,0\rangle = \frac{1}{\sqrt{2}}(|p>|n> - |n>|p>) \tag{12.45}$$

and that of an *isotriplet*

$$|1,1\rangle = |p>|p>, \tag{12.46}$$

$$|1,0\rangle = \frac{1}{\sqrt{2}}(|p>|n> + |n>|p>), \tag{12.47}$$

$$|1,-1\rangle = |n>|n>. \tag{12.48}$$

Is the deuteron an isosinglet state or an isotriplet? If it were an isotriplet ($|1,0\rangle$) we should have seen nn and pp bound states of comparable energy in nature (because of isospin symmetry), but such states do not exist. Therefore the deuteron has to be an isosinglet state ($|0,0\rangle$).

As an exercise on the implications of isospin symmetry we will consider nucleon-nucleon scattering. We will eventually be able to compute ratios of scattering cross-sections between different processes. For example, consider

$$\begin{array}{l}\text{(I) } p + p \to d + \pi^+ \\ \text{(II) } p + n \to d + \pi^0\end{array} \tag{12.49}$$

The only assumption that we put in will be that the interaction is of the form $V = \alpha\, \mathbf{I}^{(1)} \cdot \mathbf{I}^{(2)}$. The dot product here refers to the abstract isospin space. The cross-section, σ, is proportional to $|\mathcal{M}|^2$, where $\mathcal{M}$ is the scattering amplitude given by

$$\mathcal{M} = \langle \text{ final } | \;\; \alpha\, \mathbf{I}^{(1)} \cdot \mathbf{I}^{(2)} \;\; | \text{ initial } \rangle\,. \tag{12.50}$$

The initial and final states can be denoted in more detail as

$$|\text{ initial } \rangle \;=\; \left| \; I^{(i)}, I_3^{(i)}, \gamma^{(i)} \; \right\rangle, \tag{12.51}$$

$$|\text{ final } \rangle \;=\; \left| \; I^{(f)}, I_3^{(f)}, \gamma^{(f)} \; \right\rangle, \tag{12.52}$$

where $\gamma^{(i)}$ and $\gamma^{(f)}$ denote degrees of freedom other than isospin, such as the spatial dependence of the wave function and spin.

Exercise. Consider the generalization of tensor operators discussed in section 6.7 to the case of isospin. Show that $\mathbf{I}^{(1)} {\cdot} \mathbf{I}^{(2)}$ is an 'isotensor' of rank zero. Refer to exercise 6.7.5 for the spin-analogue of the same problem.

Exercise. Show that the expectation of $\mathbf{I}^{(1)} \cdot \mathbf{I}^{(2)}$ is $\frac{1}{4}$ in an isotriplet state and $-\frac{3}{4}$ in an isosinglet state.

Using (12.38) and the fact that the deuteron is an isosinglet we know that the isospins of the final states in (I) and (II) are $|1,1\rangle$ and $|1,0\rangle$, respectively. According to (12.45) and (12.48) the initial states in (I) and (II) have isospin values $|1,1\rangle$ and $(1/\sqrt{2})(|1,0\rangle + |0,0\rangle)$. We will now employ the (isospin analogue) of the Wigner-Eckart theorem (6.259) discussed in detail in sections 6.7 and 6.8 to compute the ratio of the scattering amplitudes $\mathcal{M}_{\text{I}}$ and $\mathcal{M}_{\text{II}}$. For completeness let us start by restating the Wigner-Eckart theorem (6.259) in the present context:

$$\langle I^{(f)} I_3^{(f)}, \gamma^{(f)} | T_{00} | I^{(i)} I_3^{(i)}, \gamma^{(i)} \rangle \;= \tag{12.53}$$

$$(I^{(f)} I_3^{(f)} | 00 I^{(i)} I_3^{(i)}) \;\; (-1)^{I^{(f)} - I^{(i)}} \; \frac{1}{\sqrt{2I^{(i)}+1}} \;\; \langle I^{(f)}, \gamma^{(f)} || T_0 || I^{(i)}, \gamma^{(i)} \rangle. \tag{12.54}$$

Here $T_{00} \;\equiv\; V \;=\; \alpha\, \mathbf{I}^{(1)} \cdot \mathbf{I}^{(2)}$, which is an isoscalar, as discussed in the exercise above. $(I^{(f)} I_3^{(f)} | 00 I^{(i)} I_3^{(i)})$ is a Clebsch-Gordon coefficient and $\langle I^{(f)}, \gamma^{(f)} || T_0 || I^{(i)}, \gamma^{(i)} \rangle$ is a reduced matrix element defined in the same sense as in section 6.8.
Now let us re-write more carefully the scattering amplitudes for the two processes in the light of what we have just learned

$$\mathcal{M}_{\text{I}} \;=\; \left\langle \; I^{(f)} = 1, I_3^{(f)} = 1, \gamma^{(f)} \; \middle| \; T_{00} \; \middle| \; I^{(i)} = 1, I_3^{(i)} = 1, \gamma^{(i)} \; \right\rangle \tag{12.55}$$

$$= (11|0011)\,\frac{1}{\sqrt{3}}\,\langle I^{(f)}=1,\gamma^{(f)}\,||T_{00}||\,I^{(i)}=1,\gamma^{(i)}\rangle \tag{12.56}$$

$$\mathcal{M}_{\rm II} = \frac{1}{\sqrt{2}}\left\langle\, I^{(f)}=1, I_3^{(f)}=0,\gamma^{(f)}\,\middle|\,T_{00}\,\middle|\,I^{(i)}=1, I_3^{(i)}=0,\gamma^{(i)}\,\right\rangle \tag{12.57}$$

$$+\frac{1}{\sqrt{2}}\left\langle\, I^{(f)}=1, I_3^{(f)}=0,\gamma^{(f)}\,\middle|\,T_{00}\,\middle|\,I^{(i)}=0, I_3^{(i)}=0,\gamma^{(i)}\,\right\rangle \tag{12.58}$$

$$= (10|0010)\,\frac{1}{\sqrt{3}}\,\langle I^{(f)}=1,\gamma^{(f)}\,||T_{00}||\,I^{(i)}=1,\gamma^{(i)}\rangle \tag{12.59}$$

$$+0. \tag{12.60}$$

$$\tag{12.61}$$

Note that the second term in $\mathcal{M}_{\rm II}$ vanishes due to the isospin conservation, which is also manifested by a vanishing Clebsch-Gordon prefactor. The relevant Clebsch-Gordon coefficients are easily evaluated:

$$(11|0011) \;=\; (10|0010) \;=\; 1. \tag{12.62}$$

We can now write the ratio of the scattering amplitudes:

$$\frac{\mathcal{M}_{\rm I}}{\mathcal{M}_{\rm II}} = \frac{1}{(1/\sqrt{2})}\;, \tag{12.63}$$

where common dynamical factors (which would not be as easy to compute) have dropped out thanks to the Wigner-Eckart theorem. It follows

$$\frac{\sigma_{\rm I}}{\sigma_{\rm II}} = 2, \tag{12.64}$$

which is in approximate agreement with the observed ratio. [2]

As a further example, we will consider pion-nucleon scattering. We want to compute the ratio of *total* cross-sections assuming a similar interaction as in the previous example

$$\frac{\sigma(\,\pi^+ + p\;\rightarrow\;\text{anything}\,)}{\sigma(\,\pi^- + p\;\rightarrow\;\text{anything}\,)}\;.$$

The possibilities are

$$\begin{array}{l}(\text{a})\;\pi^+ + p\;\rightarrow\;\pi^+ + p\,,\\ (\text{b})\;\pi^- + p\;\rightarrow\;\pi^- + p\,,\\ (\text{c})\;\pi^- + p\;\rightarrow\;\pi^0 + n\,.\end{array} \tag{12.65}$$

There are more exotic possibilities, involving, for example, particles with strangeness, but these are not dominant at relatively low energies.
Once again we need the isospins for the initial and final states, which are obtained by a standard Clebsch-Gordan expansion

$$\begin{array}{l}\pi^+ + p\;:\;|1,1\rangle\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle = \left|\tfrac{3}{2},\tfrac{3}{2}\right\rangle\;,\\ \pi^- + p\;:\;|1,-1\rangle\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle = \tfrac{1}{\sqrt{3}}\left|\tfrac{3}{2},-\tfrac{1}{2}\right\rangle - \sqrt{\tfrac{2}{3}}\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle\;,\\ \pi^0 + n\;:\;|1,0\rangle\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle = \sqrt{\tfrac{2}{3}}\left|\tfrac{3}{2},-\tfrac{1}{2}\right\rangle + \tfrac{1}{\sqrt{3}}\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle\;.\end{array} \tag{12.66}$$

As in the example of nucleon-meson scattering we define the relevant matrix elements

$$\begin{aligned} \mathcal{M}_{\frac{3}{2}} &= \langle\ \tfrac{3}{2}, m \mid V \mid \tfrac{3}{2}, m\ \rangle\,, \\ \mathcal{M}_{\frac{1}{2}} &= \langle\ \tfrac{1}{2}, m \mid V \mid \tfrac{1}{2}, m\ \rangle\,, \end{aligned} \tag{12.67}$$

which are independent of m. A computation similar to the previous example of nucleon-nucleon scattering yields (apart from common prefactors) the following amplitudes for the reactions in (12.65)

$$\begin{aligned} \mathcal{M}_{\mathrm{a}} &= \mathcal{M}_{\frac{3}{2}} \\ \mathcal{M}_{\mathrm{b}} &= \tfrac{1}{3}\mathcal{M}_{\frac{3}{2}} + \tfrac{2}{3}\mathcal{M}_{\frac{1}{2}} \\ \mathcal{M}_{\mathrm{c}} &= \tfrac{\sqrt{2}}{3}\mathcal{M}_{\frac{3}{2}} - \tfrac{\sqrt{2}}{3}\mathcal{M}_{\frac{1}{2}} \end{aligned} \tag{12.68}$$

Guided by empirical data we will further assume that $\mathcal{M}_{\frac{3}{2}} >> \mathcal{M}_{\frac{1}{2}}$, which leads to the following ratios for the cross-sections

$$\sigma_a : \sigma_b : \sigma_c = 9 : 1 : 2. \tag{12.69}$$

As the total cross-section is the sum of individual processes we obtain

$$\frac{\sigma(\,\pi^+ + p\,)}{\sigma(\,\pi^- + p\,)} = \frac{\sigma_a}{\sigma_b + \sigma_c} = 3 \tag{12.70}$$

again in approximate agreement with the observed value. [2]

12.3 The Eightfold Way and the flavor $SU(3)$ symmetry

The discovery of the concept of *strangeness*, mentioned in the previous section, was motivated by the existence of particles that are produced strongly but decay only weakly. For instance, K^+, which can be produced by $\pi^- + p \to K^+ + \Sigma^-$, has a lifetime which is comparable to that of π^+ albeit being more than three times heavier. Hence Gell-Mann and independently Nishijima postulated the existence of a separate quantum number, S, called strangeness, such that $S(K^+) = 1$, $S(\Sigma^-) = -1$ and $S(\pi) = S(N) = 0$, etc. It was assumed that strong interactions conserved S (at least approximately), while weak interactions did not. Hence the strangeness changing strong decays of K^+ (or Σ^-) were forbidden, giving it a higher than usual lifetime.

The classification of the newly found particles as members of some higher multiplet structure was less obvious then the case of isospin, however. Strange partners of the familiar nucleons, for example, are up to 40% heavier, making an identification of the underlying symmetry and the multiplet structure less straightforward.

In the light of the quark model, it appears an obvious generalization to add another component for an extra quark to the isospin vector space

$$u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, d = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, s = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{12.71}$$

In this case the transformations that 'rotate' the components of (12.71) into each other, while preserving the norm, have to be elements of the group $SU(3)$ (which we will investigate closely very soon). However, history followed the reverse of this path. *First* particle multiplets were identified as representations of the $SU(3)$ group, the same way nucleons and pions were identified as representations of the isospin $SU(2)$ symmetry. *Then* came the question of what the fundamental representation, as in (12.71), should correspond to, giving rise to the quark model. As quarks were never directly observed, for a period they were considered as useful bookkeeping devices without physical content.

In this perspective the flavor $SU(3)$ symmetry may appear to be mainly of historical interest. However $SU(3)$ symmetry appears in another and much more fundamental context in strong interaction physics. The quarks posses another quantum number, called *color*, which again form representations of an $SU(3)$ group. This is believed to be an *exact* symmetry of strong interactions, in fact modern theory of strong interactions is a 'gauge theory' of this color group, called *quantum chromodynamics*. (The reader is referred to section 8.3 for a brief discussion of gauge transformations). Flavor $SU(N_f)$ symmetry on the other hand, where N_f is the number quark flavors, becomes increasingly inaccurate for $N_f > 3$. The reason is that the other known quarks, namely charm, bottom (beauty) and top (truth) are significantly heavier than the hadronic energy scale. (The 'bare' mass of the *charm* quark is already heavier than the two nucleons, which set the hadronic energy scale. The bottom and top are even heavier. [2])

Before discussing the significance and the physical implications of the quark model, we will establish some mathematical preliminaries about the group $SU(3)$. In many respects it will resemble the more familiar group $SU(2)$ discussed in some detail in chapter 5, but there are a number of subtle differences. The reader shall note that most of what is being said trivially generalizes to other unitary groups, $SU(N)$, but we will stick to $N = 3$ in the following. The reader is also invited to revisit section 5.1 whenever necessary, in reference to Lie groups, Lie algebras and related concepts.

Given a complex vector, a_k, of three dimensions, we want to find those transformations

$$a_k \to U_{kl} a_l \tag{12.72}$$

that preserve the norm, $\sum_k a_k^* a_k$, of a. It is seen that such a matrix U has to satisfy the following unitary relation

$$U^\dagger = U^{-1}. \tag{12.73}$$

To verify that all such matrices form a group, we observe that

$$(UV)^\dagger = V^\dagger U^\dagger = V^{-1} U^{-1} = (UV)^{-1}, \tag{12.74}$$

for any two unitary matrices U and V. This group of 3×3 unitary matrices is denoted by $U(3)$. The unitarity relation imposes 9 constraints on the total of 18 real degrees of freedom of a 3×3 complex matrix. Hence the group $U(3)$ has 9 dimensions. Multiplying U by a phase, $e^{i\phi}$, still leaves the norm invariant. Therefore $U(3)$ can be decomposed into a direct product $U(1) \times SU(3)$ where $SU(3)$ consists of 3×3 unitary matrices of unit determinant. Because of this additional constraint $SU(3)$ has 8 dimensions. Since arbitrary phase factors are of no physical interest, it is the group

$SU(3)$ and not $U(3)$ that is of main interest. The reader is invited to compare the structure of $SU(3)$ to that of $SU(2)$ discussed in section 5.7.

As discussed in section 5.1, any unitary matrix, U, can be written in the form

$$U = e^{iH} \tag{12.75}$$

where H is a hermitian matrix. Therefore we will express elements of $SU(3)$ as

$$U = e^{i\sum_k \alpha_k \lambda_k} \tag{12.76}$$

where λ_k are 8 linearly independent matrices forming the basis of the Lie algebra of $SU(3)$. (We shall at times refer to the Lie algebra with the name of the group, the meaning being apparent from the context.) The unit determinant condition requires that all λ_k are traceless, since $\det(e^A) = e^{\mathrm{tr}A}$. An explicit basis is constructed in analogy to the Pauli algebra of spin operators

$$\begin{aligned}
\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \\
\lambda_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \\
\lambda_8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.
\end{aligned} \tag{12.77}$$

The generators, λ_k, obey the following relation

$$\mathrm{tr}(\lambda_j \lambda_k) = 2\delta_{jk}, \tag{12.78}$$

which can be verified explicitly as matrix identities.
The Lie algebra structure is given by the commutators of λ_k

$$[\lambda_j, \lambda_k] = 2if_{jkl}\lambda_l, \tag{12.79}$$

where f_{jkl} are the antisymmetric structure constants similar to the ϵ_{jkl} of $SU(2)$ given in (5.32). We can also introduce the constants, δ_{jkl}, via the anticommutator relation

$$[\lambda_j, \lambda_k]_+ = \frac{4}{3}\delta_{jk} + 2\delta_{jkl}\lambda_l, \tag{12.80}$$

This is the *fundamental* or defining representation of $SU(3)$. As in the case of $SU(2)$ higher dimensional representations obeying the same structure can be found. The fundamental relation to be preserved is (12.79), regardless of the dimension of the representation.

As it turns out the set of generators in (12.77) is not the most useful basis in the study of $SU(3)$. In the case of $SU(2)$ the identification of 'ladder' operators $\{J_+, J_-\}$ proved useful, which satisfied an 'eigenvalue equation'

$$[J_0, J_\pm] = \pm\hbar J_\pm. \tag{12.81}$$

In chapter 5, these relations have been used to construct the angular momentum spectrum as well as the function space representation of the rotation algebra, namely spherical harmonics. The generators of $SU(3)$ can be arranged into a very similar form to that of $SU(2)$. We first introduce the *F-spin* operators

$$F_i = \frac{1}{2}\lambda_i. \tag{12.82}$$

With another change of basis we arrive at the 'standard' form of the generators of the Lie algebra of $SU(3)$

$$\begin{aligned} T_\pm &= F_1 \pm iF_2, && (12.83)\\ T_0 &= F_3, && (12.84)\\ V_\pm &= F_4 \pm iF_5, && (12.85)\\ U_\pm &= F_6 \pm iF_7, && (12.86)\\ Y &= \frac{2}{\sqrt{3}}F_8. && (12.87) \end{aligned}$$

Exercise. Using the convention in (12.71) show that T_0 is the isospin operator, I_3.

Exercise. Derive the Gell-Mann–Nishijima relation (12.44), starting with the observation that $Y = B + S$. (Recall that the strange quark has $S = -1$ by convention). Y is called the *hypercharge*.

In the basis (12.87) the commutation relations between the generators can be expressed in a succinct manner. First, we have

$$[Y, T_0] = 0, \tag{12.88}$$

which defines (not uniquely) a maximal set of mutually commuting operators $\{Y, T_0\}$ and

$$\begin{aligned} [Y, T_\pm] &= 0, && (12.89)\\ [Y, U_\pm] &= \pm U_\pm, && (12.90)\\ [Y, V_\pm] &= \pm V_\pm, && (12.91) \end{aligned}$$

$$\begin{aligned} [T_0, T_\pm] &= \pm T_\pm, && (12.92)\\ [T_0, U_\pm] &= \mp\frac{1}{2}U_\pm, && (12.93)\\ [T_0, V_\pm] &= \pm\frac{1}{2}V_\pm, && (12.94) \end{aligned}$$

which relate the remaining generators $\{T_\pm, V_\pm, U_\pm\}$ to this maximal set by 'eigenvalue equations' and

$$
\begin{aligned}
[T_+, T_-] &= 2T_0, && (12.95)\\
[U_+, U_-] &= \frac{3}{2}Y - T_0 = 2U_0, && (12.96)\\
[V_+, V_-] &= \frac{3}{2}Y + T_0 = 2V_0, && (12.97)
\end{aligned}
$$

which relate commutators of generators with opposite eigenvalues to the maximal set $\{Y, T_0\}$. Note that, U_0 and V_0 are linear combinations of T_0 and Y. Finally, we have

$$
\begin{aligned}
[T_+, V_-] &= -U_-, && (12.98)\\
[T_+, U_+] &= V_+, && (12.99)\\
[U_+, V_-] &= T_-, && (12.100)\\
[T_+, V_+] &= 0, && (12.101)\\
[T_+, U_-] &= 0, && (12.102)\\
[U_+, V_+] &= 0. && (12.103)
\end{aligned}
$$

Any remaining commutators follow from hermiticity.
The same way the angular momentum ladder operators have been used to construct the representations of $SU(2)$, we will use these commutation relations to construct representations of $SU(3)$. In the case of $SU(2)$ the representations lay on a line on the J_0 axis. However, since there are two mutually commuting generators in $SU(3)$ as given in (12.88), the representations will now lie in a $T_0 - Y$-plane. The maximum number of mutually commuting generators of a Lie algebra is called its *rank*. Thus, $SU(2)$ has rank 1, while $SU(3)$ has rank 2.
When the basis of a Lie algebra is expressed in such a way to satisfy the form of the eigenvalue relations as given above, it is said to be in *Cartan-Weyl form.* This form is essential for easy labeling of the representations of the group, as the relation between the states in a given representation can be conveniently expressed in terms of ladder operators. A formal definition and a detailed discussion of the Cartan-Weyl form is beyond the scope of this chapter. The interested reader is instead referred to a very readable account given in chapter 12 of [4].

Another important property of $SU(2)$ is the existence of an operator (namely the total angular momentum, J^2) which commutes with *all* of the generators. An operator which commutes with all generators of a Lie group is called a *Casimir operator.* As in the case of J^2 and $SU(2)$, Casimir operators can be used to label irreducible representations of the Lie algebra, similar to the way it was done in section 5.5. We can construct two such independent Casimir operators for the group $SU(3)$.

$$
\begin{aligned}
C_1 &= \sum_k \lambda_k^2 = -\frac{2i}{3}\sum_{jkl} f_{jkl}\lambda_j\lambda_k\lambda_l, && (12.104)\\
C_2 &= \sum_{jkl} d_{jkl}\lambda_j\lambda_k\lambda_l && (12.105)
\end{aligned}
$$

In general the number of independent Casimir operators of a Lie group is equal to its rank.

The utility of Casimir operators arises from the fact that all states in a given representation assume the same value for a Casimir operator. This is because the states in a given representation are connected by the action of the generators of the Lie algebra and such generators commute with the Casimir operators. This property can be used to label representations in terms of the values of the Casimir operators. For example, it was shown in section 5.5 how to label the irreducible representations of the angular momentum algebra $SU(2)$ in terms of the value of the total angular momentum.

Now we will construct explicit representations of $SU(3)$. Because of (12.88) we can label states by the eigenvalues of T_0 and Y operators, $|t_3, y\rangle$:

$$T_0 \, |t_3, y\rangle = t_3 \, |t_3, y\rangle \, , \tag{12.106}$$

$$Y \, |t_3, y\rangle = y \, |t_3, y\rangle \, . \tag{12.107}$$

From the commutation relations we have presented above (the Cartan-Weyl form) we can write down the effect of various generators on the state $|t_3, y\rangle$. For example, we have

$$U_0 \, |t_3, y\rangle = (\frac{3}{4}y - \frac{1}{2}t_3) \, |t_3, y\rangle \, , \tag{12.108}$$

$$V_0 \, |t_3, y\rangle = (\frac{3}{4}y + \frac{1}{2}t_3) \, |t_3, y\rangle \, . \tag{12.109}$$

The same way that $J_\pm \, |m\rangle$ is proportional to $|m \pm 1\rangle$ in the case of the angular momentum algebra, we have

$$T_\pm \, |t_3, y\rangle = \alpha \left| t_3 \pm \frac{1}{2}, y \right\rangle , \tag{12.110}$$

$$U_\pm \, |t_3, y\rangle = \beta \left| t_3 \pm \frac{1}{2}, y \pm 1 \right\rangle , \tag{12.111}$$

$$V_\pm \, |t_3, y\rangle = \gamma \left| t_3 \mp \frac{1}{2}, y \pm 1 \right\rangle . \tag{12.112}$$

The effect of these operators to the states in the $y - t_3$ plane have been outlined in Fig. (12.1).
The representations of $SU(3)$ are constructed analogous to those of $SU(2)$ by identifying the 'boundary' states annihilated by raising (or lowering) operators. All other states of the representation are then constructed by successive application of ladder operators $T_\pm, U_\pm, V_\pm$. The representations for hexagons with sides of length p and q in the $T_0 - Y$-plane. Such a representation is labeled as $D(p, q)$ and it has a dimensionality of $\frac{1}{2}(p+1)(q+1)(p+q+2)$. Figure (12.2) shows the representation $D(2, 1)$ as an example. The details of this procedure is beyond the scope of this chapter. The interested reader is referred to [4], especially chapters 7 and 8.
As another example for the representations of $SU(3)$, the pion family forms part of an *octet* corresponding to the $D(1, 1)$ representation. The representations $D(1, 0)$ and $D(0, 1)$ correspond to the triplets of quarks and antiquarks, respectively. (See Fig. (12.3).) All other representations can be constructed by combining these two conjugate representations. For example the pion octet (or any other meson octet) is therefore realized as states of a quark - antiquark pair. A notation suggestive of the dimensionality of the representation can be used to identify representations. For example,

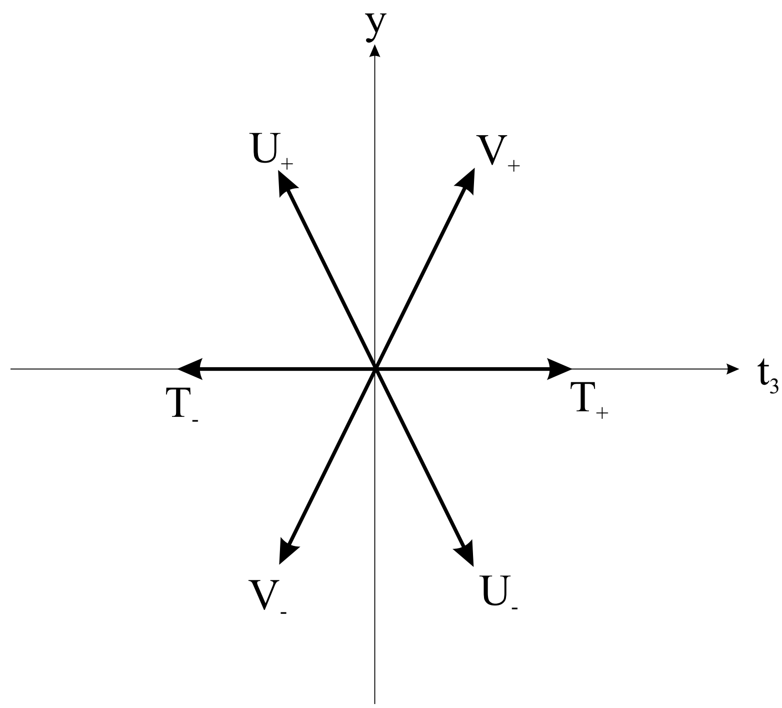

Figure 12.1: The effect of $SU(3)$ ladder operators on the $y - t_3$ plane.

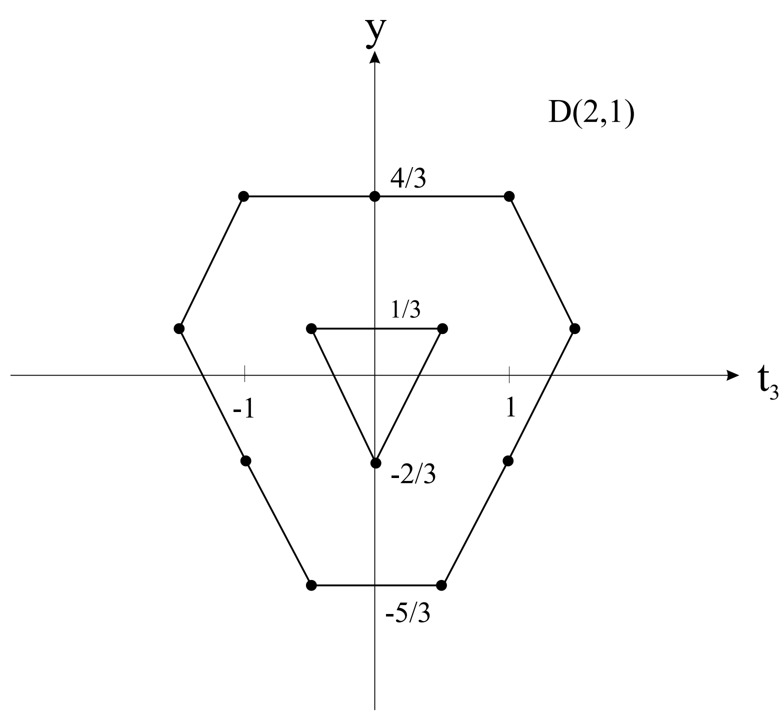

Figure 12.2: $D(2,1)$ representation of $SU(3)$. The states in the inner triangle are doubly degenerate.

$D(1,0)$ and $D(0,1)$ are denoted by [3] and $[\bar{3}]$, respectively. The octet $D(1,1)$ is written as [8] etc. This way the quark-antiquark states can be represented as follows

$$[3] \otimes [\bar{3}] = [8] \oplus [1] \tag{12.113}$$

The additional singlet state corresponds to the η' meson. This expansion can be compared, for example, to the case of adding two spin triplet states, in the case of $SU(2)$, where we would write

$$[3] \otimes [3] = [1] \oplus [3] \oplus [5]. \tag{12.114}$$

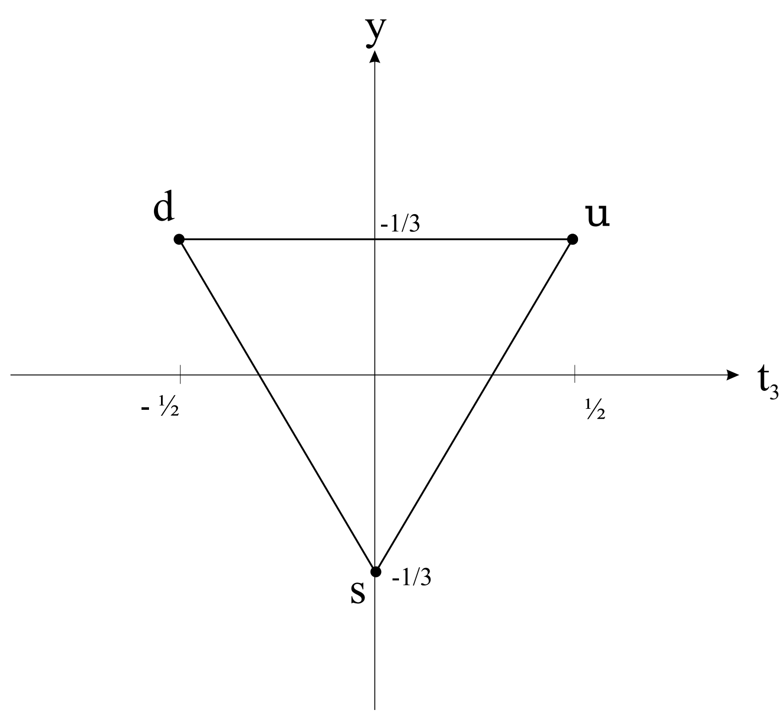

Figure 12.3: $D(1,0)$ or the fundamental representation of flavor $SU(3)$ symmetry.

The three quarks in the fundamental representation can now be written as

$$u = \left|\frac{1}{2}, \frac{1}{3}\right\rangle, \tag{12.115}$$

$$d = \left|-\frac{1}{2}, \frac{1}{3}\right\rangle, \tag{12.116}$$

$$s = \left|0, -\frac{2}{3}\right\rangle. \tag{12.117}$$

The Gell-Mann–Nishijima relation can then be succinctly expressed as

$$Q = \frac{1}{2}y + t_3, \tag{12.118}$$

from which the quark charges follow

$$Q_u = \frac{2}{3}, \tag{12.119}$$

$$Q_d = -\frac{1}{3}, \tag{12.120}$$

$$Q_s = -\frac{1}{3}. \tag{12.121}$$

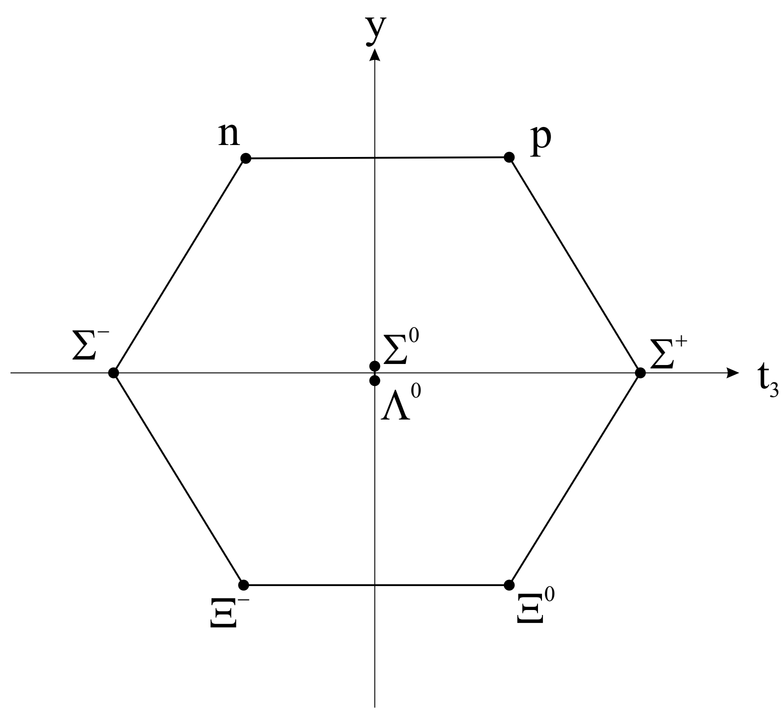

Figure 12.4: The baryon octet as a $D(1,1)$ representation of $SU(3)$.

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