Chapter 1

Lagrangian Mechanics

Our introduction to Quantum Mechanics will be based on its correspondence to Classical Mechanics. For this purpose we will review the relevant concepts of Classical Mechanics. An important concept is that the equations of motion of Classical Mechanics can be based on a variational principle, namely, that along a path describing classical motion the action integral assumes a minimal value (Hamiltonian Principle of Least Action).

1.1 Basics of Variational Calculus

The derivation of the Principle of Least Action requires the tools of the calculus of variation which we will provide now.
Definition: A *functional* $S[\,]$ is a map

$$S[\,] \,:\, \mathcal{F} \,\to \mathbb{R}\,;\; \mathcal{F} \,=\, \{\vec{q}(t);\, \vec{q}:\, [t_0, t_1] \,\subset\, \mathbb{R} \,\to\, \mathbb{R}^M;\, \vec{q}(t) \text{ differentiable}\} \tag{1.1}$$

from a space $\mathcal{F}$ of vector-valued functions $\vec{q}(t)$ onto the real numbers. $\vec{q}(t)$ is called the *trajectory* of a system of M degrees of freedom described by the *configurational coordinates* $\vec{q}(t) = (q_1(t), q_2(t), \ldots q_M(t))$.
In case of N classical particles holds $M = 3N$, i.e., there are $3N$ configurational coordinates, namely, the position coordinates of the particles in any kind of coordianate system, often in the Cartesian coordinate system. It is important to note at the outset that for the description of a classical system it will be necessary to provide information $\vec{q}(t)$ as well as $\frac{d}{dt}\vec{q}(t)$. The latter is the velocity vector of the system.
Definition: A functional $S[\,]$ is *differentiable*, if for any $\vec{q}(t) \in \mathcal{F}$ and $\delta\vec{q}(t) \in \mathcal{F}_\epsilon$ where

$$\mathcal{F}_\epsilon \,=\, \{\delta\vec{q}(t);\, \delta\vec{q}(t) \,\in \mathcal{F},\, |\delta\vec{q}(t)| \,<\, \epsilon,\, |\frac{d}{dt}\delta\vec{q}(t)| \,<\, \epsilon, \forall t,\, t \,\in\, [t_0, t_1] \subset \mathbb{R}\} \tag{1.2}$$

a functional $\delta S[\,\cdot\,,\,\cdot\,]$ exists with the properties

$$\begin{array}{ll} (i) & S[\vec{q}(t) \,+\, \delta\vec{q}(t)] \,=\, S[\vec{q}(t)] \,+\, \delta S[\vec{q}(t), \delta\vec{q}(t)] \,+\, O(\epsilon^2) \\ (ii) & \delta S[\vec{q}(t), \delta\vec{q}(t)] \text{ is linear in } \delta\vec{q}(t). \end{array} \tag{1.3}$$

$\delta S[\,\cdot\,,\,\cdot\,]$ is called the *differential* of $S[\,]$. The linearity property above implies

$$\delta S[\vec{q}(t), \alpha_1\,\delta\vec{q}_1(t) \,+\, \alpha_2\,\delta\vec{q}_2(t)] \,=\, \alpha_1\,\delta S[\vec{q}(t), \delta\vec{q}_1(t)] \,+\, \alpha_2\,\delta S[\vec{q}(t), \delta\vec{q}_2(t)]\,. \tag{1.4}$$

Note: $\delta\vec{q}(t)$ describes small variations around the trajectory $\vec{q}(t)$, i.e. $\vec{q}(t) + \delta\vec{q}(t)$ is a 'slightly' different trajectory than $\vec{q}(t)$. We will later often assume that only variations of a trajectory $\vec{q}(t)$ are permitted for which $\delta\vec{q}(t_0) = 0$ and $\delta\vec{q}(t_1) = 0$ holds, i.e., at the ends of the time interval of the trajectories the variations vanish.
It is also important to appreciate that $\delta S[\,\cdot\,,\,\cdot\,]$ in conventional differential calculus does not correspond to a differentiated function, but rather to a differential of the function which is simply the differentiated function multiplied by the differential increment of the variable, e.g., $df = \frac{df}{dx}dx$ or, in case of a function of M variables, $df = \sum_{j=1}^{M} \frac{\partial f}{\partial x_j} dx_j$.
We will now consider a particular class of functionals $S[\,]$ which are expressed through an integral over the the interval $[t_0, t_1]$ where the integrand is a function $L(\vec{q}(t), \frac{d}{dt}\vec{q}(t), t)$ of the configuration vector $\vec{q}(t)$, the velocity vector $\frac{d}{dt}\vec{q}(t)$ and time t. We focus on such functionals because they play a central role in the so-called action integrals of Classical Mechanics.
In the following we will often use the notation for velocities and other time derivatives $\frac{d}{dt}\vec{q}(t) = \dot{\vec{q}}(t)$ and $\frac{dx_j}{dt} = \dot{x}_j$.
Theorem: Let

$$S[\vec{q}(t)] \;=\; \int_{t_0}^{t_1} dt\, L(\vec{q}(t), \dot{\vec{q}}(t), t) \tag{1.5}$$

where $L(\,\cdot\,,\,\cdot\,,\,\cdot\,)$ is a function differentiable in its three arguments. It holds

$$\delta S[\vec{q}(t), \delta\vec{q}(t)] \;=\; \int_{t_0}^{t_1} dt \left\{ \sum_{j=1}^{M} \left[\frac{\partial L}{\partial q_j} \;-\; \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j} \right) \right] \delta q_j(t) \right\} \;+\; \sum_{j=1}^{M} \frac{\partial L}{\partial \dot{q}_j} \delta q_j(t) \bigg|_{t_0}^{t_1} . \tag{1.6}$$

For a proof we can use conventional differential calculus since the functional (1.6) is expressed in terms of 'normal' functions. We attempt to evaluate

$$S[\vec{q}(t) \,+\, \delta\vec{q}(t)] \;=\; \int_{t_0}^{t_1} dt\, L(\vec{q}(t) \,+ \delta\vec{q}(t), \dot{\vec{q}}(t) \,+ \delta\dot{\vec{q}}(t), t) \tag{1.7}$$

through Taylor expansion and identification of terms linear in $\delta q_j(t)$, equating these terms with $\delta S[\vec{q}(t), \delta\vec{q}(t)]$. For this purpose we consider

$$L(\vec{q}(t) \,+ \delta\vec{q}(t), \dot{\vec{q}}(t) \,+ \delta\dot{\vec{q}}(t), t) \;=\; L(\vec{q}(t), \dot{\vec{q}}(t), t) \;+\; \sum_{j=1}^{M} \left(\frac{\partial L}{\partial q_j}\delta q_j \;+\; \frac{\partial L}{\partial \dot{q}_j}\delta \dot{q}_j \right) \;+\; O(\epsilon^2) \tag{1.8}$$

We note then using $\frac{d}{dt} f(t)g(t) = \dot{f}(t)g(t) + f(t)\dot{g}(t)$

$$\frac{\partial L}{\partial \dot{q}_j}\delta \dot{q}_j \;=\; \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\delta q_j \right) \;-\; \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} \right) \delta q_j \,. \tag{1.9}$$

This yields for $S[\vec{q}(t) + \delta\vec{q}(t)]$

$$S[\vec{q}(t)] \;+\; \int_{t_0}^{t_1} dt \sum_{j=1}^{M} \left[\frac{\partial L}{\partial q_j} \;-\; \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j} \right) \right] \delta q_j \;+\; \int_{t_0}^{t_1} dt \sum_{j=1}^{M} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\delta q_j \right) \;+\; O(\epsilon^2) \tag{1.10}$$

From this follows (1.6) immediately.

We now consider the question for which functions the functionals of the type (1.5) assume extreme values. For this purpose we define
Definition: An extremal of a differentiable functional $S[\,]$ is a function $q_e(t)$ with the property

$$\delta S[\vec{q}_e(t), \delta\vec{q}(t)] \;=\; 0 \quad \text{for all } \delta\vec{q}(t) \,\in\, \mathcal{F}_\epsilon. \tag{1.11}$$

The extremals $\vec{q}_e(t)$ can be identified through a condition which provides a suitable differential equation for this purpose. This condition is stated in the following theorem.
Theorem: *Euler–Lagrange Condition*
For the functional defined through (1.5), it holds in case $\delta\vec{q}(t_0) \,=\, \delta\vec{q}(t_1) \,=\, 0$ that $\vec{q}_e(t)$ is an extremal, if and only if it satisfies the conditions ($j \,=\, 1, 2, \ldots, M$)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) \;-\; \frac{\partial L}{\partial q_j} \;=\; 0 \tag{1.12}$$

The proof of this theorem is based on the property
Lemma: If for a continuous function f(t)

$$f:\; [t_0, t_1] \subset \mathbb{R} \;\rightarrow\; \mathbb{R} \tag{1.13}$$

holds

$$\int_{t_0}^{t_1} dt\, f(t) h(t) \;=\; 0 \tag{1.14}$$

for any continuous function $h(t) \,\in\, \mathcal{F}_\epsilon$ with $h(t_0) \,=\, h(t_1) \,=\, 0$, then

$$f(t) \;\equiv\; 0 \quad \text{on } [t_0, t_1]. \tag{1.15}$$

We will not provide a proof for this Lemma.
The proof of the above theorem starts from (1.6) which reads in the present case

$$\delta S[\vec{q}(t), \delta\vec{q}(t)] \;=\; \int_{t_0}^{t_1} dt \left\{ \sum_{j=1}^{M} \left[\frac{\partial L}{\partial q_j} \;-\; \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\right] \delta q_j(t) \right\} \;. \tag{1.16}$$

This property holds for any δq_j with $\delta\vec{q}(t) \,\in\, \mathcal{F}_\epsilon$. According to the Lemma above follows then (1.12) for $j \,=\, 1, 2, \ldots M$. On the other side, from (1.12) for $j \,=\, 1, 2, \ldots M$ and $\delta q_j(t_0) \,=\, \delta q_j(t_1) \,=\, 0$ follows according to (1.16) the property $\delta S[\vec{q}_e(t), \cdot\,] \;\equiv\; 0$ and, hence, the above theorem.

An Example

As an application of the above rules of the variational calculus we like to prove the well-known result that a straight line in $\mathbb{R}^{\nvDash}$ is the shortest connection (geodesics) between two points (x_1, y_1) and (x_2, y_2). Let us assume that the two points are connected by the path $y(x)$, $y(x_1) \,=\, y_1$, $y(x_2) \,=\, y_2$. The length of such path can be determined starting from the fact that the incremental length ds in going from point $(x, y(x))$ to $(x + dx, y(x + dx))$ is

$$ds \;=\; \sqrt{(dx)^2 \,+\, (\frac{dy}{dx}dx)^2} \;=\; dx\sqrt{1 \,+\, (\frac{dy}{dx})^2}\;. \tag{1.17}$$

The total path length is then given by the integral

$$s \;=\; \int_{x_0}^{x_1} dx \sqrt{1 \,+\, (\frac{dy}{dx})^2}\;. \tag{1.18}$$

s is a functional of $y(x)$ of the type (1.5) with $L(y(x), \frac{dy}{dx}) \;=\; \sqrt{1 \,+\, (dy/dx)^2}$. The shortest path is an extremal of $s[y(x)]$ which must, according to the theorems above, obey the Euler–Lagrange condition. Using $y' \;=\; \frac{dy}{dx}$ the condition reads

$$\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) \;=\; \frac{d}{dx}\left(\frac{y'}{\sqrt{1 \,+\, (y')^2}}\right) \;=\; 0\;. \tag{1.19}$$

From this follows $y'/\sqrt{1 \,+\, (y')^2} \;=\; const$ and, hence, $y' \;=\; const$. This in turn yields $y(x) \;=\; ax \,+\, b$. The constants a and b are readily identified through the conditons $y(x_1) \;=\; y_1$ and $y(x_2) \;=\; y_2$. One obtains

$$y(x) \;=\; \frac{y_1 \,-\, y_2}{x_1 \,-\, x_2}\,(x \,-\, x_2) \,+\, y_2\;. \tag{1.20}$$

Exercise 1.1.1: Show that the shortest path between two points on a sphere are great circles, i.e., circles whose centers lie at the center of the sphere.

1.2 Lagrangian Mechanics

The results of variational calculus derived above allow us now to formulate the Hamiltonian Principle of Least Action of Classical Mechanics and study its equivalence to the Newtonian equations of motion.
Threorem: *Hamiltonian Principle of Least Action*
The trajectories $\vec{q}(t)$ of systems of particles described through the Newtonian equations of motion

$$\frac{d}{dt}(m_j\dot{q}_j) \,+\, \frac{\partial U}{\partial q_j} \;=\; 0 \quad ;\; j = 1, 2, \ldots M \tag{1.21}$$

are extremals of the functional, the so-called *action integral*,

$$S[\vec{q}(t)] \;=\; \int_{t_0}^{t_1} dt\, L(\vec{q}(t), \dot{\vec{q}}(t), t) \tag{1.22}$$

where $L(\vec{q}(t), \dot{\vec{q}}(t), t)$ is the so-called *Lagrangian*

$$L(\vec{q}(t), \dot{\vec{q}}(t), t) \;=\; \sum_{j=1}^{M} \frac{1}{2} m_j \dot{q}_j^2 \,-\, U(q_1, q_2, \ldots, q_M)\;. \tag{1.23}$$

Presently we consider only velocity–independent potentials. Velocity–dependent potentials which describe particles moving in electromagnetic fields will be considered below.

For a proof of the Hamiltonian Principle of Least Action we inspect the Euler–Lagrange conditions associated with the action integral defined through (1.22, 1.23). These conditions read in the present case

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0 \rightarrow -\frac{\partial U}{\partial q_j} - \frac{d}{dt}(m_j \dot{q}_j) = 0 \tag{1.24}$$

which are obviously equivalent to the Newtonian equations of motion.

Particle Moving in an Electromagnetic Field

We will now consider the Newtonian equations of motion for a single particle of charge q with a trajectory $\vec{r}(t) = (x_1(t), x_2(t), x_3(t))$ moving in an electromagnetic field described through the electrical and magnetic field components $\vec{E}(\vec{r}, t)$ and $\vec{B}(\vec{r}, t)$, respectively. The equations of motion for such a particle are

$$\frac{d}{dt}(m\dot{\vec{r}}) = \vec{F}(\vec{r}, t)\ ;\ \vec{F}(\vec{r}, t) = q\,\vec{E}(\vec{r}, t) + \frac{q}{c}\,\vec{v} \times \vec{B}(\vec{r}, t) \tag{1.25}$$

where $\frac{d\vec{r}}{dt} = \vec{v}$ and where $\vec{F}(\vec{r}, t)$ is the Lorentz force.
The fields $\vec{E}(\vec{r}, t)$ and $\vec{B}(\vec{r}, t)$ obey the Maxwell equations

$$\nabla \times \vec{E} + \tfrac{1}{c}\tfrac{\partial}{\partial t}\vec{B} = 0 \tag{1.26}$$

$$\nabla \cdot \vec{B} = 0 \tag{1.27}$$

$$\nabla \times \vec{B} - \tfrac{1}{c}\tfrac{\partial}{\partial t}\vec{E} = \frac{4\pi \vec{J}}{c} \tag{1.28}$$

$$\nabla \cdot \vec{E} = 4\pi\rho \tag{1.29}$$

where $\rho(\vec{r}, t)$ describes the charge density present in the field and $\vec{J}(\vec{r}, t)$ describes the charge current density. Equations (1.27) and (1.28) can be satisfied implicitly if one represents the fields through a scalar potential $V(\vec{r}, t)$ and a vector potential $\vec{A}(\vec{r}, t)$ as follows

$$\vec{B} = \nabla \times \vec{A} \tag{1.30}$$

$$\vec{E} = -\nabla V - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}\,. \tag{1.31}$$

Gauge Symmetry of the Electromagnetic Field

It is well known that the relationship between fields and potentials (1.30, 1.31) allows one to transform the potentials without affecting the fields and without affecting the equations of motion (1.25) of a particle moving in the field. The transformation which leaves the fields invariant is

$$\vec{A}'(\vec{r}, t) = \vec{A}(\vec{r}, t) + \nabla K(\vec{r}, t) \tag{1.32}$$

$$V'(\vec{r}, t) = V(\vec{r}, t) - \frac{1}{c}\frac{\partial}{\partial t}K(\vec{r}, t) \tag{1.33}$$

Lagrangian of Particle Moving in Electromagnetic Field

We want to show now that the equation of motion (1.25) follows from the Hamiltonian Principle of Least Action, if one assumes for a particle the Lagrangian

$$L(\vec{r},\dot{\vec{r}},t) \;=\; \frac{1}{2}m\vec{v}^2 \;-\; q\,V(\vec{r},t) \;+\; \frac{q}{c}\,\vec{A}(\vec{r},t)\cdot\vec{v}\;. \tag{1.34}$$

For this purpose we consider only one component of the equation of motion (1.25), namely,

$$\frac{d}{dt}(mv_1) \;=\; F_1 \;=\; -q\,\frac{\partial V}{\partial x_1} \;+\; \frac{q}{c}\,[\vec{v}\times\vec{B}]_1\;. \tag{1.35}$$

We notice using (1.30), e.g., $B_3 = \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}$

$$[\vec{v}\times\vec{B}]_1 \;=\; \dot{x}_2 B_3 \;-\; \dot{x}_3 B_2 \;=\; \dot{x}_2\left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right) \;-\; \dot{x}_3\left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}\right)\;. \tag{1.36}$$

This expression allows us to show that (1.35) is equivalent to the Euler–Lagrange condition

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) \;-\; \frac{\partial L}{\partial x_1} \;=\; 0\;. \tag{1.37}$$

The second term in (1.37) is

$$\frac{\partial L}{\partial x_1} \;=\; -q\,\frac{\partial V}{\partial x_1} \;+\; \frac{q}{c}\left(\frac{\partial A_1}{\partial x_1}\,\dot{x}_1 \;+\; \frac{\partial A_2}{\partial x_1}\,\dot{x}_2 \;+\; \frac{\partial A_3}{\partial x_1}\,\dot{x}_3\right)\;. \tag{1.38}$$

The first term in (1.37) is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) \;=\; \frac{d}{dt}(m\dot{x}_1) \;+\; \frac{q}{c}\,\frac{dA_1}{dt} \;=\; \frac{d}{dt}(m\dot{x}_1) \;+\; \frac{q}{c}\left(\frac{\partial A_1}{\partial x_1}\,\dot{x}_1 \;+\; \frac{\partial A_1}{\partial x_2}\,\dot{x}_2 \;+\; \frac{\partial A_1}{\partial x_3}\,\dot{x}_3\right)\;. \tag{1.39}$$

The results (1.38, 1.39) together yield

$$\frac{d}{dt}(m\dot{x}_1) \;=\; -q\,\frac{\partial V}{\partial x_1} \;+\; \frac{q}{c}\,O \tag{1.40}$$

where

$$\begin{aligned} O \;&=\; \frac{\partial A_1}{\partial x_1}\,\dot{x}_1 \;+\; \frac{\partial A_2}{\partial x_1}\,\dot{x}_2 \;+\; \frac{\partial A_3}{\partial x_1}\,\dot{x}_3 \;-\; \frac{\partial A_1}{\partial x_1}\,\dot{x}_1 \;-\; \frac{\partial A_1}{\partial x_2}\,\dot{x}_2 \;-\; \frac{\partial A_1}{\partial x_3}\,\dot{x}_3 \\ &=\; \dot{x}_2\left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right) \;-\; \dot{x}_3\left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}\right) \end{aligned} \tag{1.41}$$

which is identical to the term (1.36) in the Newtonian equation of motion. Comparing then (1.40, 1.41) with (1.35) shows that the Newtonian equations of motion and the Euler–Lagrange conditions are, in fact, equivalent.

1.3 Symmetry Properties in Lagrangian Mechanics

Symmetry properties play an eminent role in Quantum Mechanics since they reflect the properties of the elementary constituents of physical systems, and since these properties allow one often to simplify mathematical descriptions.
We will consider in the following two symmetries, gauge symmetry and symmetries with respect to spatial transformations.
The gauge symmetry, encountered above in connection with the transformations (1.32, 1.33) of electromagnetic potentials, appear in a different, surprisingly simple fashion in Lagrangian Mechanics. They are the subject of the following theorem.
Theorem: *Gauge Transformation of Lagrangian*
The equation of motion (Euler–Lagrange conditions) of a classical mechanical system are unaffected by the following transformation of its Lagrangian

$$L'(\vec{q}, \dot{\vec{q}}, t) \;=\; L(\vec{q}, \dot{\vec{q}}, t) \;+\; \frac{d}{dt}\frac{q}{c}\,K(\vec{q}, t) \tag{1.42}$$

This transformation is termed gauge transformation. The factor $\frac{q}{c}$ has been introduced to make this transformation equivalent to the gauge transformation (1.32, 1.33) of electyromagnetic potentials. Note that one adds the *total* time derivative of a function $K(\vec{r}, t)$ the Lagrangian. This term is

$$\frac{d}{dt}K(\vec{r}, t) \;=\; \frac{\partial K}{\partial x_1}\,\dot{x}_1 \;+\; \frac{\partial K}{\partial x_2}\,\dot{x}_2 \;+\; \frac{\partial K}{\partial x_3}\,\dot{x}_3 \;+\; \frac{\partial K}{\partial t} \;=\; (\nabla K)\cdot\vec{v} \;+\; \frac{\partial K}{\partial t}\;. \tag{1.43}$$

To prove this theorem we determine the action integral corresponding to the transformed Lagrangian

$$\begin{aligned} S'[\vec{q}(t)] \;&=\; \int_{t_0}^{t_1} dt L'(\vec{q}, \dot{\vec{q}}, t) \;=\; \int_{t_0}^{t_1} dt L(\vec{q}, \dot{\vec{q}}, t) \;+\; \frac{q}{c}\,K(\vec{q}, t)\Big|_{t_0}^{t_1} \\ &=\; S[\vec{q}(t)] \;+\; \frac{q}{c}\,K(\vec{q}, t)\Big|_{t_0}^{t_1} \end{aligned} \tag{1.44}$$

Since the condition $\delta\vec{q}(t_0) \;=\; \delta\vec{q}(t_1) \;=\; 0$ holds for the variational functions of Lagrangian Mechanics, Eq. (1.44) implies that the gauge transformation amounts to adding a constant term to the action integral, i.e., a term not affected by the variations allowed. One can conclude then immediately that any extremal of $S'[\vec{q}(t)]$ is also an extremal of $S[\vec{q}(t)]$.
We want to demonstrate now that the transformation (1.42) is, in fact, equivalent to the gauge transformation (1.32, 1.33) of electromagnetic potentials. For this purpose we consider the transformation of the single particle Lagrangian (1.34)

$$L'(\vec{r}, \dot{\vec{r}}, t) \;=\; \frac{1}{2}m\vec{v}^2 \;-\; q\,V(\vec{r}, t) \;+\; \frac{q}{c}\,\vec{A}(\vec{r}, t)\cdot\vec{v} \;+\; \frac{q}{c}\frac{d}{dt}K(\vec{r}, t)\;. \tag{1.45}$$

Inserting (1.43) into (1.45) and reordering terms yields using (1.32, 1.33)

$$\begin{aligned} L'(\vec{r}, \dot{\vec{r}}, t) \;&=\; \frac{1}{2}m\vec{v}^2 \;-\; q\left(V(\vec{r}, t) \;-\; \frac{1}{c}\frac{\partial K}{\partial t}\right) \;+\; \frac{q}{c}\left(\vec{A}(\vec{r}, t) \;+\; \nabla K\right)\cdot\vec{v} \\ &=\; \frac{1}{2}m\vec{v}^2 \;-\; q\,V'(\vec{r}, t) \;+\; \frac{q}{c}\,\vec{A}'(\vec{r}, t)\cdot\vec{v}\;. \end{aligned} \tag{1.46}$$

Obviously, the transformation (1.42) corresponds to replacing in the Lagrangian potentials $V(\vec{r},t)$, $\vec{A}(\vec{r},t)$ by gauge transformed potentials $V'(\vec{r},t)$, $\vec{A}'(\vec{r},t)$. We have proven, therefore, the equivalence of (1.42) and (1.32, 1.33).
We consider now invariance properties connected with coordinate transformations. Such invariance properties are very familiar, for example, in the case of central force fields which are invariant with respect to rotations of coordinates around the center.
The following description of spatial symmetry is important in two respects, for the connection between invariance properties and constants of motion, which has an important analogy in Quantum Mechanics, and for the introduction of infinitesimal transformations which will provide a crucial method for the study of symmetry in Quantum Mechanics. The transformations we consider are the most simple kind, the reason being that our interest lies in achieving familiarity with the principles (just mentioned above) of symmetry properties rather than in providing a general tool in the context of Classical Mechanics. The transformations considered are specified in the following definition.
Definition: Infinitesimal One-Parameter Coordinate Transformations
A *one-parameter coordinate transformation* is decribed through

$$\vec{r}\,' \;=\; \vec{r}\,'(\vec{r},\epsilon)\;,\; \vec{r},\,\vec{r}\,' \,\in\, \mathbb{R}^{\mathbb{3}}\;,\; \epsilon\,\in\,\mathbb{R} \tag{1.47}$$

where the origin of ϵ is chosen such that

$$\vec{r}\,'(\vec{r},0) \;=\; \vec{r}\;. \tag{1.48}$$

The corresponding *infinitesimal transformation* is defined for small ϵ through

$$\vec{r}\,'(\vec{r},\epsilon) \;=\; \vec{r} \,+\, \epsilon\,\vec{R}(\vec{r}) \,+\, O(\epsilon^2)\,; \quad \vec{R}(\vec{r}) \;=\; \left.\frac{\partial \vec{r}\,'}{\partial \epsilon}\right|_{\epsilon=0} \tag{1.49}$$

In the following we will denote *unit vectors* as $\hat{a}$, i.e., for such vectors holds $\hat{a}\cdot\hat{a} \;=\; 1$.

Examples of Infinitesimal Transformations

The beauty of infinitesimal transformations is that they can be stated in a very simple manner. In case of a *translation transformation* in the direction $\hat{e}$ nothing new is gained. However, we like to provide the transformation here anyway for later reference

$$\vec{r}\,' \;=\; \vec{r} \,+\, \epsilon\,\hat{e}\;. \tag{1.50}$$

A non-trivial example is furnished by the infinitesimal rotation around axis $\hat{e}$

$$\vec{r}\,' \;=\; \vec{r} \,+\, \epsilon\,\hat{e}\times\vec{r}\;. \tag{1.51}$$

We would like to derive this transformation in a somewhat complicated, but nevertheless instructive way considering rotations around the x_3–axis. In this case the transformation can be written in matrix form

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} \;=\; \begin{pmatrix} \cos\epsilon & -\sin\epsilon & 0 \\ \sin\epsilon & \cos\epsilon & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{1.52}$$

In case of small ϵ this transformation can be written neglecting terms $O(\epsilon^2)$ using $\cos\epsilon = 1 + O(\epsilon^2)$, $\sin\epsilon = \epsilon + O(\epsilon^2)$

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 & -\epsilon & 0 \\ \epsilon & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + O(\epsilon^2) . \tag{1.53}$$

One can readily verify that in case $\hat{e} = \hat{e}_3$ ($\hat{e}_j$ denoting the unit vector in the direction of the x_j–axis) (1.51) reads

$$\vec{r}' = \vec{r} - x_2\,\hat{e}_1 + x_1\,\hat{e}_2 \tag{1.54}$$

which is equivalent to (1.53).
Anytime, a classical mechanical system is invariant with respect to a coordinate transformation a constant of motion exists, i.e., a quantity $C(\vec{r},\dot{\vec{r}})$ which is constant along the classical path of the system. We have used here the notation corresponding to single particle motion, however, the property holds for any system.
The property has been shown to hold in a more general context, namely for fields rather than only for particle motion, by Noether. We consider here only the 'particle version' of the theorem. Before the embark on this theorem we will comment on what is meant by the statement that a classical mechanical system is invariant under a coordinate transformation. In the context of Lagrangian Mechanics this implies that such transformation leaves the Lagrangian of the system unchanged.
Theorem: *Noether's Theorem*
If $L(\vec{q},\dot{\vec{q}},t)$ is invariant with respect to an infinitesimal transformation $\vec{q}\,' = \vec{q} + \epsilon\,\vec{Q}(\vec{q})$, then $\sum_{j=1}^{M} Q_j \frac{\partial L}{\partial \dot{x}_j}$ is a constant of motion.
We have generalized in this theorem the definition of infinitesimal coordinate transformation to M–dimensional vectors $\vec{q}$.
In order to prove Noether's theorem we note

$$q'_j = q_j + \epsilon\,Q_j(\vec{q}) \tag{1.55}$$

$$\dot{q}'_j = \dot{q}_j + \epsilon \sum_{k=1}^{M} \frac{\partial Q_j}{\partial q_k}\,\dot{q}_k . \tag{1.56}$$

Inserting these infinitesimal changes of q_j and $\dot{q}_j$ into the Lagrangian $L(\vec{q},\dot{\vec{q}},t)$ yields after Taylor expansion, neglecting terms of order $O(\epsilon^2)$,

$$L'(\vec{q},\dot{\vec{q}},t) = L(\vec{q},\dot{\vec{q}},t) + \epsilon \sum_{j=1}^{M} \frac{\partial L}{\partial q_j} Q_j + \epsilon \sum_{j,k=1}^{M} \frac{\partial L}{\partial \dot{q}_j}\frac{\partial Q_j}{\partial q_k}\dot{q}_k \tag{1.57}$$

where we used $\frac{d}{dt}Q_j = \sum_{k=1}^{M}(\frac{\partial}{\partial q_k}Q_j)\dot{q}_k$. Invariance implies $L' = L$, i.e., the second and third term in (1.57) must cancel each other or both vanish. Using the fact, that *along the classical path* holds the Euler-Lagrange condition $\frac{\partial L}{\partial q_j} = \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_j})$ one can rewrite the sum of the second and third term in (1.57)

$$\sum_{j=1}^{M}\left(Q_j \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) + \frac{\partial L}{\partial \dot{q}_j}\frac{d}{dt}Q_j \right) = \frac{d}{dt}\left(\sum_{j=1}^{M} Q_j \frac{\partial L}{\partial \dot{q}_j} \right) = 0 \tag{1.58}$$

From this follows the statement of the theorem.

Application of Noether's Theorem

We consider briefly two examples of invariances with respect to coordinate transformations for the Lagrangian $L(\vec{r}, \vec{v}) = \frac{1}{2} m \vec{v}^2 - U(\vec{r})$.
We first determine the constant of motion in case of invariance with respect to translations as defined in (1.50). In this case we have $Q_j = \hat{e}_j \cdot \hat{e}$, $j = 1, 2, 3$ and, hence, Noether's theorem yields the constant of motion ($q_j = x_j$, $j = 1, 2, 3$)

$$\sum_{j=1}^{3} Q_j \frac{\partial L}{\partial \dot{x}_j} = \hat{e} \cdot \sum_{j=1}^{3} \hat{e}_j m \dot{x}_j = \hat{e} \cdot m\vec{v} \,. \tag{1.59}$$

We obtain the well known result that in this case the momentum in the direction, for which translational invariance holds, is conserved.
We will now investigate the consequence of rotational invariance as described according to the infinitesimal transformation (1.51). In this case we will use the same notation as in (1.59), except using now $Q_j = \hat{e}_j \cdot (\hat{e} \times \vec{r})$. A calculation similar to that in (1.59) yields the constant of motion $(\hat{e} \times \vec{r}) \cdot m\vec{v}$. Using the cyclic property $(\vec{a} \times \vec{b}) \cdot \vec{c} = (\vec{b} \times \vec{c}) \cdot \vec{a} = (\vec{c} \times \vec{a}) \cdot \vec{b}$ allows one to rewrite the constant of motion $\hat{e} \cdot (\vec{r} \times m\vec{v})$ which can be identified as the component of the angular momentum $m\vec{r} \times \vec{v}$ in the $\hat{e}$ direction. It was, of course, to be expected that this is the constant of motion.
The important result to be remembered for later considerations of symmetry transformations in the context of Quantum Mechanics is that it is sufficient to know the consequences of infinitesimal transformations to predict the symmetry properties of Classical Mechanics. It is not necessary to investigate the consequences of global. i.e, not infinitesimal transformations.