Solution to Problem Set 9
Physics 480 / Fall 1999
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Problem 1: An important relationship

1a. Since $A = diag(\lambda_1, \lambda_2, \lambda_3)$ implies $A^n = diag(\lambda_1^n, \lambda_2^n, \lambda_3^n)$

$U = exp(iA) = diag(e^{i\lambda_1}, e^{i\lambda_2}, e^{i\lambda_3})$ and $detU = exp(i(\lambda_1 + \lambda_2 + \lambda_3))$ Hence $detU = 1$ implies $(\lambda_1 + \lambda_2 + \lambda_3) = 0 mod 2\pi$ i.e. $tr(A) = 0 mod 2\pi$.

1b. Let h(1), h(2), h(3) be the 3 eigenvectors of A. i.e. $\sum_{m=1}^{3} A_m^n h^m(i) = \epsilon_i h^n(i), n = 1, 2, 3$ Orthogonality of the eigenvectors give,

$$\sum_m h_m(i) h^m(j) = \delta_{ij}$$

Let us define the matrix R as

$$T_k^m = h_k(m)$$

It follows from the above orthogonality condition that

$$(T^{-1})_m^l = h^l(m)$$

Let us consider the matrix

$$A' = TAT^{-1}$$

The matrix elements of A' are

$$\begin{aligned}
(A')_n^m &= T_k^m A_L^k (T^{-1})_n^l \\
&= h_k(m) A_l^k h^l(n) \\
&= h_k(m) \epsilon_n h^k(n) \\
&= \epsilon_n \delta_{mn}
\end{aligned}$$

Hence A' is diagonal.

1c.

$$\begin{aligned}
TUT^{-1} &= Te^{iA}T^{-1} \\
&= 1 \sum_{n=0}^{\infty} \frac{i^n}{n!} TA^nT^{-1} \\
&= \sum_{n=0}^{\infty} \frac{i^n}{n!} TAT^{-1}TAT^{-1} ......TAT^{-1} \\
&= exp[iTAT^{-1}]
\end{aligned}$$

1d.

$$B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

implies $B^n = \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix}$

Hence,$e^B = \begin{pmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{pmatrix}$

1e.

$$\begin{aligned} Tr(AB) &= A_{ij}B_{ji} \\ &= B_{ji}A_{ij} \\ &= Tr(BA) \end{aligned}$$

Therefore, $Tr(TAT^{-1}) = Tr(T^{-1}TA) = Tr(A)$

1f.

$$U = e^{iA}$$

$$TUT^{-1} = e^{iTAT^{-1}}$$

$$detTUT^{-1} = det(e^{iTAT^{-1}})$$

but, $detTUT^{-1} = det(U)$ therefore, $det(U) = 1$ implies

$$det(e^{iTAT^{-1}}) = 1$$

which implies

$$Tr(TAT^{-1}) = 0mod2\pi$$

But since $Tr(AB) = Tr(BA)$

$Tr(TAT^{-1}) = 0$ imples $Tr(A) = 0mod2\pi$

Problem 2: Spin along x_1 axis

2a. Taylor expanding the operator(3) we get

$$e^{i\theta\sigma_1/2} = \sum_{n=0}^{\infty} \frac{i^n}{n!}(\frac{\theta_1}{2})^n \sigma_1^n$$

Using the fact that

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$\sigma_1^2 = I$ and $i^2 = -1$ we get,

$$e^{i\theta\sigma_1/2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n!}(\frac{\theta_1}{2})^{2n}.I + i\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(\frac{\theta_1}{2})^{2n+1}.\sigma_1$$

$$\begin{aligned} e^{i\theta\sigma_1/2} &= \cos\frac{\theta_1}{2}I + i\sin\frac{\theta_1}{2}\sigma_1 \\ &= \begin{pmatrix} \cos\frac{\theta_1}{2} & i\sin\frac{\theta_1}{2} \\ i\sin\frac{\theta_1}{2} & \cos\frac{\theta_1}{2} \end{pmatrix} \end{aligned}$$

2b. The required probability is given by

$$P = |M|^2$$

where,

$$M = y^T AX$$

where

$$y = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$A = \begin{pmatrix} \cos(\theta_1/2) & i\sin(\theta_1/2) \\ i\sin(\theta_1/2) & \cos(\theta_1/2) \end{pmatrix}$$

$$X = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

It follows $M = \frac{1}{\sqrt{2}}(\cos(\theta_1/2) + i\sin(\theta_1/2))$ Therefore, $P = 1/2$

Problem 3: Spin in a magnetic field

If the axis of quantization is chosen to be along the magnetic field, then

$$H = -\frac{e}{2mc}\vec{\sigma}.\vec{B}$$

$$H = -\frac{e}{2mc}B_0\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The time dependence of an arbitrary spin state $\begin{pmatrix} a \\ b \end{pmatrix}$ is given by

$$\begin{aligned} \psi(t) &= e^{-iHt}\psi(0) \\ &= \begin{pmatrix} e^{i\frac{e}{2mc}Bt} & a \\ e^{-i\frac{e}{2mc}Bt} & b \end{pmatrix} \end{aligned}$$

which is the same as a rotation of the spin about the z-axis by an angle $\frac{eBt}{2mc}$

3b. For $\psi(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\psi(t) = \begin{pmatrix} e^{i\frac{e}{2mc}Bt} \\ 0 \end{pmatrix}$$

Problem 4: Spin Dynamics

4a.

$$H = \frac{g\mu_B\hbar}{2}(B_x\sigma_x + B_y\sigma_y + B_z\sigma_z) + a(\frac{\hbar}{2})^2(\sigma_x^e\sigma_x^n + \sigma_y^e\sigma_y^n + \sigma_z^e\sigma_z^n)$$

Using the fact that

$$\sigma_x|\pm >= |\mp >$$

$$\sigma_y|\pm >= \pm i|\mp >$$

$$\sigma_z|\pm >= \pm 1|\pm >$$

for both electronic and nuclear spins, we get the Hamiltonian in a matrix form in the given basis ($\hbar = 1$)

$$H = \begin{pmatrix} \frac{g\mu_B}{2}B_3 + \frac{a}{4} & 0 & \frac{g\mu_B}{2}(B_1 - iB_2) & 0 \\ 0 & \frac{g\mu_B}{2}B_3 - \frac{a}{4} & \frac{a}{2} & \frac{g\mu_B}{2}(B_1 - iB_2) \\ \frac{g\mu_B}{2}(B_1 + iB_2) & \frac{a}{2} & -\frac{g\mu_B}{2}B_3 - \frac{a}{4} & 0 \\ 0 & \frac{g\mu_B}{2}(B_1 + iB_2) & 0 & -\frac{g\mu_B}{2}B_3 + \frac{a}{4} \end{pmatrix}$$

For $\vec{B} = (0, 0, b)$ the eigenvalues are $\frac{a}{4} \pm 8.8b, -\frac{a}{4} \pm \frac{1}{2}\sqrt{a^2 + (g\mu_B B_3)^2}$

4c. The time evolution of a wave function is given by

$$\psi(t) = e^{-iHt}\psi(0)$$

Here $\psi(0) = |2>$ We expand it in the basis of eigenvectors of H

$$|2 >= \sum_{n=1}^{4} C_n|n >$$

Therefore

$$|\psi(t) >= \sum_n C_n e^{-iE_nt}|n >$$

$$< 2|\psi(t) >= \sum_n C_n^* C_n e^{-iE_n t}$$

Therefore, Probability of finding it in another state,

$$P = 1 - | < 2|\psi(t) > |^2$$