Problem Set 9
Physics 480 / Fall 1999
Professor Klaus Schulten

Problem 1: An important relationship

Prove that for 3×3-matrices U and A, where A is also hermitean, holds the property that $U = \exp(iA)$ and $\det U = 1$ implies $\mathrm{trace}(A) = 0$. Follow the following route:

(a) Show that the stated property is true in case that A has diagonal form.

(b) Argue that a transformation T exists such that TAT^{-1} is diagonal.

(c) Argue that $TUT^{-1} = \exp(iTAT^{-1})$.

(d) Argue that for

$$B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{1}$$

follows

$$\exp(B) = \begin{pmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{pmatrix} \tag{2}$$

(e) Show that $\mathrm{trace}(AB) = \mathrm{trace}(BA)$ and, hence, $\mathrm{trace}(TAT^{-1}) = \mathrm{trace}(A)$.

(f) Use the properties above to complete the prove of the above statement.

Problem 2: Spin along x_1-axis

(a) Following an analogous derivation in class determine

$$\exp(i\theta_1\sigma_1/2) \tag{3}$$

(b) Using (a) determine the probability that a spin state observed to be in the "up" direction along the x_3-axis is found in the "up" direction along the x_2-axis.

Problem 3: Spin in a magnetic field

The spin of an electron in a (time-independent) magnetic field $\vec{B}$ is subject to the interaction $(e/2m_e\,c)\sigma \cdot \vec{B}$ where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ (vector of Pauli matrices).

(a) Argue why the time dependence of the state of the spin can be described as a rotation of the initial spin state around an axis given by $\vec{B}$.

(b) Determine the spin state of the electron at time t when the electron is initially in the state $|\frac{1}{2}, +\frac{1}{2}\rangle$.

Problem 4: Spin Dynamics of Electron in Hydrogen Atom

The spin (operator $\vec{S}$ of the electron in the hydrogen atom electronic ground state is governed by the Hamiltonian

$$H \;=\; g\mu_B \vec{B}\cdot\vec{S} \;+\; a\vec{S}\cdot\vec{I} \tag{4}$$

where the first term represents the Zeeman interaction with an external magnetic field $\vec{B} = \{b_1, B_2, B_3\}$ and the second term the hyperfine interaction with a nuclear spin-$\frac{1}{2}$ described by the the nuclear spin operators $I_j = \frac{\hbar}{2}\sigma_j$, i.e., an operator like the electron spin operator, except that it acts on the nuclear spins states $|\frac{1}{2}, \pm\frac{1}{2}\rangle$. When the spins are expressed in units $\hbar$, and B in Gauss then the relevant numerical constants are

$$g\mu_B/\hbar \;=\; 17.59B/\mu s\;, \quad a/\hbar \;=\; 8800/\mu s\;. \tag{5}$$

(a) Determine the Hamiltonian in the basis

$$\begin{aligned}
|1\rangle &= |\frac{1}{2},\frac{1}{2}\rangle_e|\frac{1}{2},\frac{1}{2}\rangle_n \\
|2\rangle &= |\frac{1}{2},\frac{1}{2}\rangle_e|\frac{1}{2},-\frac{1}{2}\rangle_n \\
|3\rangle &= |\frac{1}{2},-\frac{1}{2}\rangle_e|\frac{1}{2},\frac{1}{2}\rangle_n \\
|4\rangle &= |\frac{1}{2},-\frac{1}{2}\rangle_e|\frac{1}{2},-\frac{1}{2}\rangle_n
\end{aligned} \tag{6}$$

where $|\frac{1}{2}, \pm\frac{1}{2}\rangle_e$ denotes teh electron spin states and $|\frac{1}{2}, \pm\frac{1}{2}\rangle_n$ the nuclear spin state.

Determine the stationary states. Plot the energy of the stationary states as a function of B.

(c) Assume that the system is initially in state $|2\rangle$. What is the probability of finding it in any of the other states at a later time t in fields of 0, 200, 400, ... 2000 Gauss.

The problem set needs to be handed in by Tuesday,November 23. The web page of Physics 480 is at
`http://www.ks.uiuc.edu/Services/Class/PHYS480/`