Solutions to Problem Set 6/Problem 2
Physics 480 / Fall 1999
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Problem2
(a) in region D_j

$$-\frac{\hbar^2}{2m}\bigtriangledown^2\,\Psi(x,y)=E\,\Psi(x,y)$$

assume $\Psi(x,y)=X(x)Y(y)$

$$\Longrightarrow -\frac{\hbar^2}{2m}(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2})X(x)Y(y)=EX(x)Y(y)$$

$$\Longrightarrow -\frac{\hbar^2}{2m}(YX^{''}+Y^{''}X)=EX(x)Y(y)$$

$$\Longrightarrow -\frac{\hbar^2}{2m}(\frac{X^{''}}{X}+\frac{Y^{''}}{Y})=E$$

Since E is a constant, which requires

$$\begin{cases} -\frac{\hbar^2}{2m}\frac{X^{''}}{X} &= E_1 \\ -\frac{\hbar^2}{2m}\frac{Y^{''}}{Y} &= E_2 \end{cases}$$

where $E_1+E_2=E$. The equations above will lead to

$$\begin{cases} -\frac{\hbar^2}{2m}X^{''} &= E_1X \\ -\frac{\hbar^2}{2m}Y^{''} &= E_2Y \end{cases}$$

They are the same as 1-dimentional quantum well wave equation. Therefore we have

$$X(x)=\begin{cases} \sqrt{\frac{2}{H_j}}sin(\frac{n_1\pi x}{H_j}) & 0<x<H_j \\ 0 & \text{elsewhere} \end{cases}$$

$$Y(y)=\begin{cases} \sqrt{\frac{2}{L_j}}sin(\frac{n_2\pi y}{L_j}) & 0<y<L_j \\ 0 & \text{elsewhere} \end{cases}$$

$$\Longrightarrow \Psi(x,y)=\begin{cases} \sqrt{\frac{4}{H_jL_j}}sin(\frac{n_1\pi x}{H_j})sin(\frac{n_2\pi y}{L_j}) & \begin{pmatrix} 0<x<H_j \\ 0<y<L_j \end{pmatrix} \\ 0 & \text{elsewhere} \end{cases}$$

$$E=E_1+E_2=\frac{\hbar^2\pi^2n_1^2}{2mH_j^2}+\frac{\hbar^2\pi^2n_2^2}{2mL_j^2}$$

(b)

$$\Psi(x,y,t=0) = \begin{cases} \sqrt{\frac{4}{H_1 L_1}} sin(\frac{n_1 \pi x}{H_1}) sin(\frac{n_2 \pi y}{L_1}) & \begin{pmatrix} 0 < x < H_1 \\ 0 < y < L_1 \end{pmatrix} \\ 0 & \text{elsewhere} \end{cases}$$

(c)
As we know

$$E = \frac{\hbar^2 \pi^2}{2m} (\frac{n_1^2}{H_j^2} + \frac{n_2^2}{L_j^2})$$

In region D_1
for $n_1 = 2, n_2 = 1$

$$E = \frac{\hbar^2 \pi^2}{2m} (\frac{4}{64} + \frac{1}{400}) * 10^{20} = \frac{\hbar^2 \pi^2}{2m} * 0.065 * 10^{20}$$

for $n_1 = 3, n_2 = 1$

$$E = \frac{\hbar^2 \pi^2}{2m} (\frac{9}{64} + \frac{1}{400}) * 10^{20} = \frac{\hbar^2 \pi^2}{2m} * 0.143 * 10^{20}$$

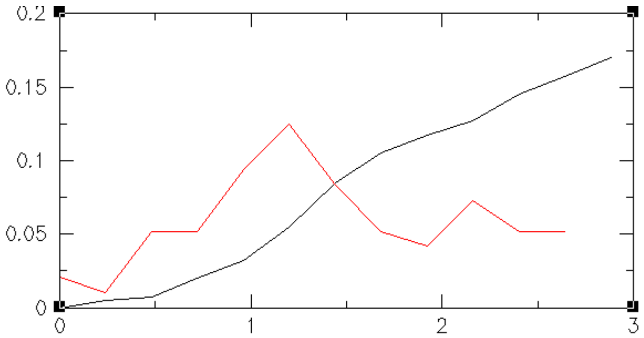
In region D_2(Ground state $n_2 = 1$)

$$E = \frac{\hbar^2 \pi^2}{2m} (\frac{n_2^2}{L_2^2}) = \frac{\hbar^2 \pi^2}{2m} (\frac{1}{3.75^2}) * 10^{20} = \frac{\hbar^2 \pi^2}{2m} * 0.071 * 10^{20}$$

The energy of the ground state in region D_2 does not vanish, which will prevent waves in D_1 propagate to D_3. The energy of the initial state with $n_1 = 3, n_2 = 1$ is higher than the barrier which makes it easy to cross region D_2, while the energy of $n_1 = 2, n_2 = 1$ is less than the barrier energy which makes it very hard to cross region D_2.

In figure 4 we can get data of $P(D_3)$ of t, then we can obtain $K(t) \equiv \frac{dP(D_3)}{dt}$ by hand or using some software(e.g *xmgr* which is available in ews machines).

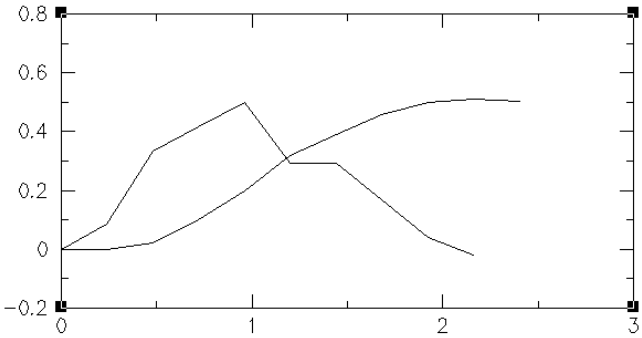
The data I use for $(n_1 = 2, n_2 = 1)$ is



Status

set#	n	stat	type	X/Y	min	at	max	at	mean	stddev
0	13	on	XY	X	0	1	2.88	13	1.44	0.898
				Y	0	1	0.17	13	0.079038	0.059739
1	12	on	XY	X	0	1	2.64	12	1.32	0.82849
				Y	0.010417	2	0.125	6	0.059028	0.029869

Figure 1: $K(t)$ as a function of $t(n_1 = 2, n_2 = 1)$ The mean value of $K(t)$ is $0.059028 * 10^{15} s^{-1}$



Status

set#	n	stat	type	X/Y	min	at	max	at	mean	stddev
0	11	on	XY	X	0	1	2.4	11	1.2	0.75895
				Y	0	1	0.51	10	0.27318	0.20465
1	10	on	XY	X	0	1	2.16	10	1.08	0.68935
				Y	-0.020833	10	0.5	5	0.21042	0.17292

Figure 2: $K(t)$ as a function of $t(n_1 = 3, n_2 = 1)$

$t(*10^{-15}s)$	$P(D_3)$
0	0
0.24	0.005
0.48	0.0075
0.72	0.020
0.96	0.0325
1.2	0.055
1.44	0.085
1.68	0.105
1.92	0.1175
2.16	0.1275
2.4	0.145
2.64	0.1575
2.88	0.170

The data I use for $(n_1 = 3, n_2 = 1)$ is

$t(*10^{-15}s)$	$P(D_3)$
0	0
0.24	0
0.48	0.02
0.72	0.10
0.96	0.20
1.2	0.32
1.44	0.39
1.68	0.46
1.92	0.50
2.16	0.51
2.4	0.505
2.64	0.51

In the figure 1 and figure 2 I reproduce the $P(D_3)$ as a function of t, then I draw $K(t)$. Note the mean value of $K(t)$ is $0.059028 * 10^{15}s^{-1}$ when $n_1 = 2, n_2 = 1$. This result will be used in next part
.

(d)

$$K = \nu\ exp(-\frac{2}{\hbar}\int_{x_1=H_1}^{x_2=H_1+H_2} \sqrt{2m(U-E)}dx_1)$$

U and E have been calculated in previous part

$$U = \frac{\hbar^2\pi^2}{2m} * 0.071 * 10^{20}$$

$$E = \frac{\hbar^2\pi^2}{2m} * 0.065 * 10^{20}$$

$$\Longrightarrow U - E = \frac{\hbar^2\pi^2}{2m} * 0.006 * 10^{20}$$

Since $\nu = \frac{n_1\hbar\pi}{2mH_1^2}$, plug in everything, finally we will get

$$K = 8.11976 * 10^{13}(s^{-1})$$

In part (c) we have got $K = 5.9028 * 10^{13}s^{-1}$, the difference is :

$$\frac{8.11976 - 5.9028}{8.11976} * 100\% = 27.3\%$$