Solutions to Problem Set 6/Problem 1
Physics 480 / Fall 1999
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5.1.1

(a) {1,i,-1,-i}
It forms a group since it satisfies the difinition of Group
{1,-1} and {1} form subgroups.

(b) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
It forms a group.
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ form subgroups.

(c) {rotation by 0^0, rotation by 90^0, rotation by 180^0,rotation by 270^0 } It forms a group.
{rotation by 0^0, rotation by 180^0 } and {rotation by 0^0 } form subgroups.

The homomorphic mapping is shown below:
1 $\longleftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \longleftrightarrow$ rotation by 0^0
i $\longleftrightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \longleftrightarrow$ rotation by 90^0
-1 $\longleftrightarrow \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \longleftrightarrow$ rotation by 180^0
-i $\longleftrightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \longleftrightarrow$ rotation by 270^0

5.1.2

$$\begin{aligned}
L_1 &= \lim_{v_1 \to 0} v_1^{-1} (R(\vec{v} = (v_1, 0, 0)^T) - 1) \\
&= \lim_{v_1 \to 0} v_1^{-1} \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & cosv_1 & -sinv_1 \\ 0 & sinv_1 & cosv_1 \end{pmatrix} - 1 \right] \\
&= \lim_{v_1 \to 0} v_1^{-1} \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -v_1 \\ 0 & v_1 & 1 \end{pmatrix} - 1 \right] \\
&= \lim_{v_1 \to 0} v_1^{-1} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -v_1 \\ 0 & v_1 & 0 \end{pmatrix}
\end{aligned}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

5.1.3

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} L_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} L_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} [L_1, L_2] &= L_1 L_2 - L_2 L_1 \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= L_3 \end{aligned}$$

$\Longrightarrow [L_1, L_2] = L_3,\ [L_2, L_1] = -L_3$
Similarly, we can prove

$$[L_2, L_3] = L_1,\ [L_3, L_2] = -L_1$$

$$[L_3, L_1] = L_2,\ [L_1, L_3] = -L_2$$

$\Longrightarrow [L_k, L_l] = \varepsilon_{klm} L_m$

5.2.1
If $\rho_{\prime}(R(\vec{v}))\Psi \equiv \Psi(R(\vec{v})\vec{r})$
then

$$\begin{aligned} &\rho_{\prime}(R(\vec{v_1})R(\vec{v_2}))\Psi(\vec{r}) \\ &= \Psi(R(\vec{v_1})R(\vec{v_2})\vec{r}) \\ &= \rho_{\prime}(R(\vec{v_1}))\Psi(R(\vec{v_2})\vec{r}) \\ &= \rho_{\prime}(R(\vec{v_2}))\rho_{\prime}(R(\vec{v_1}))\Psi(\vec{r}) \end{aligned}$$

i.e. $\rho_{\prime}(AB) = \rho_{\prime}(B)\rho_{\prime}(A)$

5.2.2 (a)

$$-\frac{i}{\hbar}J_1 = \lim_{v_1\to 0} v_1^{-1}(R(\vec{v} = (v_1, 0, 0)^T) - 1)$$

$$\begin{aligned}
-\frac{i}{\hbar}J_1 f(\vec{r}) &= \lim_{v_1\to 0} v_1^{-1}[R(\vec{v} = (v_1, 0, 0)^T) f(\vec{r}) - f(\vec{r})] \\
&= \lim_{v_1\to 0} v_1^{-1}[f(R(-v_1, 0, 0)\vec{r}) - f(\vec{r})]
\end{aligned}$$

It is easy to see

$$\begin{aligned}
R(-v_1, 0, 0)\vec{r} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & v_1 \\ 0 & -v_1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\
&= \begin{pmatrix} x_1 \\ x_2 + x_3 v_1 \\ -v_1 x2 + x_3 \end{pmatrix}
\end{aligned}$$

$\Longrightarrow f(R(-v_1, 0, 0)\vec{r}) - f(\vec{r})$

$$\begin{aligned}
&= f(\vec{r}) + v_1 x_3 \frac{\partial f}{\partial x_2} - v_1 x_2 \frac{\partial f}{\partial x_3} - f(\vec{r}) \\
&= v_1(x_3 \partial_2 f - x_2 \partial_3 f)
\end{aligned}$$

$\Longrightarrow -\frac{i}{\hbar}J_1 f(\vec{r}) = x_3\partial_2 f - x_2 \partial_3 f$
$\Longrightarrow -\frac{i}{\hbar}J_1 = x_3\partial_2 - x_2\partial_3$
(b)

$$-\frac{i}{\hbar}J_1 = x_3\partial_2 - x_2\partial_3$$

$$-\frac{i}{\hbar}J_2 = x_1\partial_3 - x_3\partial_1$$

$$-\frac{i}{\hbar}J_3 = x_2\partial_1 - x_1\partial_2$$

$$\begin{aligned}
[J_2, J_3] &= (i\hbar)^2[x_1\partial_3 - x_3\partial_1, x_2\partial_1 - x_1\partial_2] \\
&= (i\hbar)^2([x_1\partial_3, x_2\partial_1] + [x_3\partial_1, x_1\partial_2]) \\
&= (i\hbar)^2(-x_2\partial_3 + x_3\partial_2) \\
&= i\hbar J_1
\end{aligned}$$

Similarly we can get $[J_3, J_1] = i\hbar J_2$

5.2.3 It can be known from notes(5.87) that:

$$J_3 = -i\hbar\frac{\partial}{\phi}$$

then we have

$$exp(\frac{i\alpha}{\hbar}J_3)\Psi(\phi)$$

$$\begin{aligned}
&= exp(\alpha\frac{\partial}{\partial\phi})\Psi(\phi) \\
&= \Psi + \alpha\frac{\partial}{\partial\phi} + \frac{\alpha^2}{2!}\frac{\partial^2}{\partial\phi^2}\Psi + \frac{\alpha^3}{3!}\frac{\partial^3}{\partial\phi^3}\Psi + \cdots \\
&= \Psi(\phi + \alpha)
\end{aligned}$$