Problem Set 2
Physics 480 / Fall 1999
Professor Klaus Schulten

Consider a harmonic oscillator of mass m and frequency ω which is driven by some external force $F(\tau)$; the contribution of this force to the Lagrangian is $V(\tau) = -xF(\tau)$.

(a) State the Lagrangian $L(x, \dot{x}, \tau)$ of the system and derive the classical equation of motion.

(b) Show that the classical path $x_{cl}(\tau)$ with endpoints $x_{cl}(\tau = t_0) = x_0$ and $x_{cl}(\tau = t) = x$ is given by

$$x_{cl}(\tau) = \frac{x_0 \sin\omega(t-\tau) + x \sin\omega(\tau - t_0)}{\sin\omega(t-t_0)} - \frac{1}{m\omega}\int_{t_0}^{t} ds\, g(\tau, s)\, F(s) \ , \qquad (1)$$

where

$$g(\tau, s) = \begin{cases} \frac{\sin\omega(t-\tau)\sin\omega(s-t_0)}{\sin\omega(t-t_0)} & \text{for} \quad s \le \tau \\ \frac{\sin\omega(t-s)\sin\omega(\tau-t_0)}{\sin\omega(t-t_0)} & \text{for} \quad s > \tau \end{cases} \qquad (2)$$

<u>Hint</u>: In order to solve the equation of motion obtained at (a) you can follow the following steps

1. *Define the complex function $\xi = \dot{x} + i\omega x$ and prove that it satisfies the following equation of motion*

$$\dot{\xi} = i\omega\xi + \frac{F(\tau)}{m} \ .$$

2. *Show that the general solution for the above differential equation can be written as*

$$\xi(\tau) = \left[\xi_0 + \frac{1}{m}\int_{t_0}^{\tau} ds\, F(s) e^{-i\omega s}\right] e^{i\omega\tau} \ ,$$

where ξ_0 is a complex integration constant.

3. *Obtain the general solution $x(\tau)$ through the formula $x = \frac{1}{\omega} Im\{\xi\}$.*

(c) Show that the propagator $\phi(x, t|x_0, t_0)$ defined through the path integral

$$\phi(x, t|x_0, t_0) = \iint_{x(t_0)=x_0}^{x(t)=x} d[x(\tau)] \, \exp\left\{\frac{i}{\hbar} S[x(\tau)]\right\} \qquad (3)$$

for our system is given by

$$\phi(x, t|x_0, t_0) = \left[\frac{m\omega}{2\pi i\hbar \sin\omega(t-t_0)}\right]^{1/2} \exp\left\{\frac{i}{\hbar} S[x_{cl}(\tau)]\right\} \ . \qquad (4)$$

(d) Using the result for $x_{cl}(\tau)$ at (b), prove that the classical action integral is given by

$$\begin{aligned} S[x_{cl}(\tau)] &= \frac{m\omega}{2\sin\omega(t-t_0)}[(x^2+x_0^2)\cos\omega(t-t_0)-2xx_0]+ \\ & x\int_{t_0}^{t} ds\, F(s)\frac{\sin\omega(s-t_0)}{\sin\omega(t-t_0)} + x_0\int_{t_0}^{t} ds\, F(s)\frac{\sin\omega(t-s)}{\sin\omega(t-t_0)} - \\ & \frac{1}{2m\omega}\int_{t_0}^{t} d\tau \int_{t_0}^{t} ds\, F(\tau)\, g(\tau,s)\, F(s)\ . \end{aligned} \tag{5}$$

Let us apply now the results obtained so far to study the time evolution of the quantum state of a harmonic oscillator under a constant force F_0 which is suddenly turned on at $\tau = t_0 = 0$, i.e., the external force $F(\tau)$ is given by

$$F(\tau) = \begin{cases} 0 & \text{for } \tau < 0 \\ F_0 & \text{for } \tau > 0\ . \end{cases} \tag{6}$$

(e) Show that in this particular case the classical action integral (5) reads

$$S[x_{cl}(\tau)] = \frac{m\omega}{2\sin\omega t}\{[(x-a)^2+(x_0-a)^2]\cos\omega t - 2(x-a)(x_0-a)\} + \frac{m\omega^2 a^2 t}{2}\ , \tag{7}$$

where

$$a = \frac{F_0}{m\omega^2}\ . \tag{8}$$

Give a physical interpretation of the obtained result.

(f) Assuming that for $\tau < t_0 = 0$ the oscillator is in its ground state (a stationary state corresponding to the lowest possible energy value $E_0 = \frac{\hbar\omega}{2}$) described by the wave function

$$\Psi_0(x,\tau) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \exp\left(-\frac{m\omega}{2\hbar}x^2\right) e^{-i\frac{\omega\tau}{2}}\ , \tag{9}$$

prove that for $\tau = t > 0$ the oscillator will propagate into the state

$$\begin{aligned} \Psi(x,t) &= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \exp\left[-i\frac{\omega t}{2}(1-\frac{m\omega a^2}{\hbar})\right] \exp\left\{-\frac{m\omega}{2\hbar}(x-a)^2 - \right. \\ & \left. \frac{m\omega a}{\hbar}e^{-i\omega t}(x-a) + \frac{im\omega a^2}{2\hbar}\sin\omega t e^{-i\omega t}\right\} \exp\left(-\frac{m\omega a^2}{2\hbar}\right) \end{aligned} \tag{10}$$

At this stage it is convenient to introduce dimensionless quantities for further calculations. This can be done, for example, by redefining the time and length units as follows

$$T = \frac{2\pi}{\omega}\ , \tag{11}$$

and respectively

$$L = \sqrt{\frac{\hbar}{m\omega}} \ . \tag{12}$$

Now we can switch to dimensionless quantities in our expressions by simply setting $T = 1$ (i.e., $\omega = 2\pi$) and $L = 1$ (i.e., $\hbar = m\omega$). Obviously, we can recover the original quantities from the dimensionless ones at any time by multiplying them with the appropriate unit factors L and T.

(g) Show, by using dimensionless quantities, that for $t > 0$ the probability density of finding the oscillator in the spatial interval $(x, x + dx)$ is given by the following expression

$$|\Psi(x,t)|^2 = \frac{1}{\sqrt{\pi}} \exp[-(x - a + a\cos 2\pi t)^2] \ . \tag{13}$$

(h) Derive the solution of the analogue classical problem, i.e., a harmonic oscillator initially at rest and subjected to the force (6). Compare this solution to the quantum mechanical probability (13).

Since for $t > 0$ the Lagrangian $L(x, \dot{x}, t)$, in fact, is time independent one can describe the resulting state (10) through stationary states for a harmonic oscillator corresponding to a potential $V(x) = \frac{1}{2}m\omega^2x^2 - F_0x$ and introduce for such description the stationary states charcterized by discrete energy values E_n^F, $(n = 0, 1, \ldots)$ and wave functions

$$\Psi_n^F(x,t) = \psi_n^F(x)e^{-i\frac{E_n^F}{\hbar}t} \ . \tag{14}$$

(i) Prove that the ground state of the system, for $t > 0$, is described by the wave function

$$\Psi_0^F(x,t) = (\pi)^{-\frac{1}{4}} \exp\left[-\frac{1}{2}(x-a)^2\right] e^{-i\pi t(1-a^2)} \ , \tag{15}$$

where

$$a = \frac{F_0}{m\omega^2} \ . \tag{16}$$

Give a physical interpretation for a.

(j) Prove that the probability to find the considered oscillator in its ground state for $t > 0$ is given by

$$P_0 = \left| \int_{-\infty}^{\infty} dx [\Psi_0^F(x,t)]^* \Psi(x,t) \right|^2 = \exp(-\frac{a^2}{2}) \ . \tag{17}$$

(k) Show that the stationary states $\psi_n^F(x)$ can be expressed in terms of the Hermite polynomials $H_n(x)$ as follows

$$\psi_n^F(x,t) = \pi^{-\frac{1}{4}} \frac{1}{2^{n/2}\sqrt{n!}} e^{-\frac{(x-a)^2}{2}} H_n(x-a) \ . \tag{18}$$

(l) Prove that the probability of finding the oscillator at $t > 0$ in its n^{th} excited state $\Psi_n^F(x,t)$ is given by

$$P_n = \frac{(a^2/2)^n}{n!} e^{-(a^2/2)} \ , \qquad n = 0, 1, 2, \dots \ , \tag{19}$$

and check that $\sum_{n=0}^{\infty} P_n = 1$ holds.

In your derivation you may find useful the following two formulae

$$\int_{-\infty}^{\infty} dx e^{-x^2} H_n(x) H_m(x) = 2^n n! \sqrt{\pi} \delta_{n,m} \ ,$$

and

$$e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n \ .$$

(m) Using `Mathematica` or another program of your choice make the following plots

(i) P_n as function of n for $a \in \{0.5, 1.0, 1.5, 2.0\}$ and $n \leq 5$;

(ii) P_0, P_1, P_2 and P_3 as a function of $a \in [0, 5]$.

Comment on the behavior of the plotted quantities.

(n) For which value(s) of the external force F_0 will the chance to find the oscillator, for $t > 0$, in its second excited state be larger than the probability to find the oscillator in any other state?

The problem set needs to be handed in by Thursday, Sept. 23. The web page of Physics 480 is at
`http://www.ks.uiuc.edu/Services/Class/PHYS480/`