Linear Harmonic Oscillator

The linear harmonic oscillator is described by the Schrödinger equation

$$i\hbar\,\partial_t\,\psi(x,t) \;=\; \hat{H}\,\psi(x,t) \tag{1}$$

for the Hamiltonian

$$\hat{H} \;=\; -\,\frac{\hbar^2}{2m}\,\frac{\partial^2}{\partial x^2} \;+\; \frac{1}{2} m\,\omega^2 x^2\;. \tag{2}$$

It comprises one of the most important examples of elementary Quantum Mechanics. There are several reasons for its pivotal role. The linear harmonic oscillator describes vibrations in molecules and their counterparts in solids, the phonons. Many more physical systems can, at least approximately, be described in terms of linear harmonic oscillator models. However, the most eminent role of this oscillator is its linkage to the boson, one of the conceptual building blocks of microscopic physics. For example, bosons describe the modes of the electromagnetic field, providing the basis for its quantization. The linear harmonic oscillator, even though it may represent rather non-elementary objects like a solid and a molecule, provides a window into the most elementary structure of the physical world. The most likely reason for this connection with fundamental properties of matter is that the harmonic oscillator Hamiltonian (2) is symmetric in momentum and position, both operators appearing as quadratic terms.

In the present section we approach the harmonic oscillator in the framework of the Schrödinger equation. The Schrödinger equation approach will allow us to emphasize the algebraic aspects of quantum theory. This Section will be the first in which an algebraic formulation will assume center stage. Due to the pedagodical nature of these notes we will link carefully the algebraic treatment with the differential equation methods used so far in studying the Schrödinger equation description of quantum systems.

In the following we consider first the stationary states of the linear harmonic oscillator and later consider the propagator which describes the time evolution of any initial state. The stationary states of the harmonic oscillator have been considered already. In the framework of the Schrödinger equation the stationary states are solutions of (1) of the form $\psi(x,t) = \exp(-iEt/\hbar)\phi_E(x,t)$ where

$$\hat{H}\,\phi_E(x) \;=\; E\,\phi_E(x)\;. \tag{3}$$

Due to the nature of the harmonic potential which does not allow a particle with finite energy to move to arbitrary large distances, all stationary states of the harmonic oscillator must be bound states and, therefore, the natural boundary conditions apply

$$\lim_{x \to \pm\infty} \phi_E(x) \; = \; 0 \; . \tag{4}$$

Equation (3) can be solved for any $E \in \mathbf{R}$, however, only for a discrete set of E values can the boundary conditions (4) be satisfied. In the following algebraic solution of (3) we restrict the Hamiltonian $\hat{H}$ and the operators appearing in the Hamiltonian from the outset to the space of functions

$$\mathcal{N}_1 \; = \; \{ f : \mathbf{R} \to \mathbf{R}, \, f \in C_\infty, \, \lim_{x \to \pm\infty} f(x) \; = \; 0 \} \tag{5}$$

where C_∞ denotes the set of functions which together with all of their derivatives are continuous. It is important to keep in mind this restriction of the space, in which the operators used below, act. We will point out explicitly where assumptions are made which built on this restriction. If this restriction would not apply and all functions $f : \mathbf{R} \to \mathbf{R}$ would be admitted, the spectrum of $\hat{H}$ in (3) would be continuous and the eigenfunctions $\phi_E(x)$ would not be normalizable.

1 Creation and Annihilation Operators

The Hamiltonian operator (2) can be expressed in terms of the two operators

$$\hat{p} \; = \; \frac{\hbar}{i} \frac{d}{dx} \, , \quad \hat{x} \; = \; x \tag{6}$$

the first being a differential operator and the second a multiplicative operator. The operators act on the space of functions $\mathcal{N}_1$ defined in (5). The Hamiltonian $\hat{H}$ can be expressed in terms of the operators acting on the space (5)

$$\hat{H} \; = \; \frac{1}{2m} \hat{p}^2 \; + \; \frac{1}{2} m \, \omega^2 \, \hat{x}^2 \tag{7}$$

which is why these operators are of interest to us.

The cardinal property exploited below, beside the representation (7) of the Hamiltonian, is the commutation property

$$[\hat{p}, \, \hat{x}] \; = \; \frac{\hbar}{i} \, \hat{I} \tag{8}$$

which holds for any position and momentum operator. This property states that $\hat{p}$ and $\hat{x}$ obey an algebra in which the two do not commute, however, the commutator has a simple form. In order to derive (8) we recall that all operators act on functions in $\mathcal{N}_1$ and, hence, the action of $[\hat{p}, \hat{x}]$ on such functions must be considered. One obtains

$$[\hat{p}, \hat{x}]\, f(x) \;=\; \frac{\hbar}{i}\frac{d}{dx}\, x\, f(x) \;-\; x\,\frac{\hbar}{i}\frac{d}{dx}\, f(x) \;=\; \frac{\hbar}{i}\, f(x) \;=\; \frac{\hbar}{i}\,\hat{I}\, f(x)\;. \tag{9}$$

From this follows (8).

Our next step is an attempt to factorize the Hamiltonian (7) assuming that the factors are easier to handle than the Hamiltonian in yielding spectrum and eigenstates. Being guided by the identity for scalar numbers

$$(b - ic)(b + ic) \;=\; b^2 \;-\; i(cb - bc) \;+\; c^2 \;=\; b^2 \;+\; c^2 \tag{10}$$

we define

$$\hat{a}^+ \;=\; \sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} \;-\; \frac{i}{\sqrt{2m\hbar\omega}}\,\hat{p}\;, \quad \hat{a}^- \;=\; \sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} \;+\; \frac{i}{\sqrt{2m\hbar\omega}}\,\hat{p}\;. \tag{11}$$

The reader may note that we have attempted, in fact, to factor $\hat{H}/\hbar\omega$. Since $\hat{a}^+$ and $\hat{a}^-$ are operators and not scalars we cannot simply expect that the identy (10) holds for $\hat{a}^+$ and $\hat{a}^-$ since $[\hat{a}^-, \hat{a}^+] = \hat{a}^-\hat{a}^+ - \hat{a}^+\hat{a}^-$ does not necessarily vanish. In fact, the commutator property (8) implies

$$[\hat{a}^-, \hat{a}^+] \;=\; \hat{I}\;. \tag{12}$$

To prove this commutation property we determine using (11)

$$\hat{a}^-\hat{a}^+ \;=\; \frac{m\omega}{2\hbar}\,\hat{x}^2 \;+\; \frac{1}{2m\hbar\omega}\,\hat{p}^2 \;+\; \frac{i}{2\hbar}\,[\hat{p}, \hat{x}]\;. \tag{13}$$

(7) and (8) yield

$$\hat{a}^-\hat{a}^+ \;=\; \frac{1}{\hbar\omega}\,\hat{H} \;+\; \frac{1}{2}\,\hat{I}\;. \tag{14}$$

Similarly one can show

$$\hat{a}^+\hat{a}^- \;=\; \frac{1}{\hbar\omega}\,\hat{H} \;-\; \frac{1}{2}\,\hat{I}\;. \tag{15}$$

(14) and (15) together lead to the commutation property (12).

Before we continue we like to write (14, 15) in a form which will be useful below

$$\hat{H} \;=\; \hbar\omega\,\hat{a}^-\hat{a}^+ \;-\; \frac{\hbar\omega}{2}\,\hat{I} \tag{16}$$

$$\hat{H} \;=\; \hbar\omega\,\hat{a}^+\hat{a}^- \;+\; \frac{\hbar\omega}{2}\,\hat{I}\;. \tag{17}$$

We also express $\hat{a}^+$ and $\hat{a}^-$ directly in terms of the coordinate x and the differential operator $\frac{d}{dx}$

$$\hat{a}^+ \;=\; \sqrt{\frac{m\omega}{2\hbar}}\,x \;-\; \sqrt{\frac{\hbar}{2m\omega}}\,\frac{d}{dx}\;, \quad \hat{a}^- \;=\; \sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} \;+\; \sqrt{\frac{\hbar}{2m\omega}}\,\frac{d}{dx}\;. \tag{18}$$

It is of interest to note that the operators $\hat{a}^+$ and $\hat{a}^-$ are real differential operators.

Relationship between a^+ and a^-

The operators $\hat{a}^+$ and $\hat{a}^-$ are related to each other by the following property which holds for all functions $f,\, g \in \mathcal{N}_1$

$$\int_{-\infty}^{+\infty} dx\, f(x)\, a^+\, g(x) \;=\; \int_{-\infty}^{+\infty} dx\, g(x)\, a^-\, f(x)\;. \tag{19}$$

This property states that the operators $\hat{a}^+$ and $\hat{a}^-$ are the adjoints of each other.

In the following we will determine the spectrum of $\hat{H}$ and its eigenstates. The derivation will be based solely on the properties (12, 16, 17, 19).

$\hat{a}^+$ and $\hat{a}^-$ as Ladder Operators

The operators $\hat{a}^+$ and $\hat{a}^-$ allow one to generate all stationary states of a harmonic oscillator once one such state $\phi_E(x)$

$$\hat{H}\,\phi_E(x) \;=\; E\,\phi_E(x) \tag{20}$$

is available. In fact, one obtains using (16, 17, 20)

$$\begin{aligned}
\hat{H}\,\hat{a}^-\,\phi_E(x) \;&=\; (\,\hbar\omega\,\hat{a}^-\hat{a}^+ \;-\; \frac{\hbar\omega}{2}\,\hat{I}\,)\,\hat{a}^-\,\phi_E(x) \\
&=\; \hat{a}^-\,(\,\hbar\omega\,\hat{a}^+\hat{a}^- \;+\; \frac{\hbar\omega}{2}\,\hat{I} \;-\; \hbar\omega\,\hat{I}\,)\,\phi_E(x) \\
&=\; \hat{a}^-\,(\,\hat{H} \;-\; \hbar\omega\,)\,\phi_E(x) \\
&=\; (\,E \;-\; \hbar\omega\,)\,\hat{a}^-\,\phi_E(x)\;.
\end{aligned} \tag{21}$$

Similarly one can show using again (16, 17, 20)

$$\begin{aligned}
\hat{H}\,\hat{a}^+\,\phi_E(x) \;&=\; (\,\hbar\omega\,\hat{a}^+\hat{a}^- \;+\; \frac{\hbar\omega}{2}\,\hat{I}\,)\,\hat{a}^+\,\phi_E(x) \\
&=\; \hat{a}^+\,(\,\hbar\omega\,\hat{a}^-\hat{a}^+ \;-\; \frac{\hbar\omega}{2}\,\hat{I} \;+\; \hbar\omega\,\hat{I}\,)\,\phi_E(x) \\
&=\; \hat{a}^+\,(\,\hat{H} \;+\; \hbar\omega\,)\,\phi_E(x) \\
&=\; (\,E \;+\; \hbar\omega\,)\,\hat{a}^+\,\phi_E(x)\;.
\end{aligned} \tag{22}$$

Together, it holds that for a stationary state $\phi_E(x)$ of energy E defined through (20) $\hat{a}^-\phi_E(x)$ is a stationary state of energy $E - \hbar\omega$ and $\hat{a}^+\phi_E(x)$ is a stationary state of energy $E + \hbar\omega$.

The results (21, 22) can be generalized to m-fold application of the operators $\hat{a}^+$ and $\hat{a}^-$

$$\begin{aligned}
\hat{H}\,\left(\hat{a}^+\right)^m\,\phi_E(x) \;&=\; (\,E \;+\; m\,\hbar\omega\,)\,\left(\hat{a}^+\right)^m\,\phi_E(x) \\
\hat{H}\,\left(\hat{a}^-\right)^m\,\phi_E(x) \;&=\; (\,E \;-\; m\,\hbar\omega\,)\,\left(\hat{a}^-\right)^m\,\phi_E(x)\;.
\end{aligned} \tag{23}$$

One can use these relationships to construct an infinite number of stationary states by stepping up and down the spectrum in steps of $\pm\hbar\omega$. For obvious reasons one refers to $\hat{a}^+$ and $\hat{a}^-$ as ladder operators. Another name is creation ($\hat{a}^+$) and annihilation ($\hat{a}^-$) operators since these operators create and annihilate vibrational quanta $\hbar\omega$. There arise, however, two difficulties: (i) one needs to know at least one stationary state to start the construction; (ii) the construction appears to yield energy eigenvalues without lower bounds when, in fact, one expects that $E = 0$ should be a lower bound. It turns out that both difficulties can be resolved together. In fact, a state $\phi_0(x)$ which obeys the property

$$\hat{a}^- \, \phi_0(x) \; = \; 0 \tag{24}$$

on one side would lead to termination of the sequence $E_0 + m$, $m \in \mathbf{Z}$ when m is decreased, on the other side such a state is itself an eigenstate of $\hat{H}$ as can be shown using (17)

$$\hat{H} \, \phi_0(x) \; = \; (\, \hbar\omega \, \hat{a}^+ \hat{a}^- \;\; + \; \frac{\hbar\omega}{2} \, \hat{I} \,) \, \phi_0(x) \; = \; \frac{1}{2} \hbar\omega \, \phi_0(x) \; . \tag{25}$$

Of course, the solution $\phi_0(x)$ of (24) needs to be normalizable in order to represent a bound state of the harmonic oscillator, i.e., $\phi_0(x)$ should be an element of the function space $\mathcal{N}_1$ defined in (5).

The fact that $\hat{H}$ is Hermitian (is equal to its adjoint $\hat{H}\dagger$) has an important consequence for the stationary states $\phi_E(x)$. Let $\phi_E(x)$ and $\phi_{E'}(x)$ be two normalized stationary states corresponding to two *different* energies E, E', $E \neq E'$. For $f(x) = \phi_E(x)$ and $g(x) = \phi_{E'}(x)$ we have

$$0 \; = \; \langle \phi_E | \hat{H} \phi_{E'} \rangle_{\Omega_\infty} \; - \; \overline{\langle \phi_{E'} | \hat{H} \phi_E \rangle_{\Omega_\infty}} \; = \; (\, E' \; - \; E \,) \, \langle \phi_E | \phi_{E'} \rangle_{\Omega_\infty} \; . \tag{26}$$

Since $E \neq E'$ one can conclude

$$\langle \phi_E | \phi_{E'} \rangle_{\Omega_\infty} \; = \; 0 \; . \tag{27}$$

2 Ground State of the Harmonic Oscillator

A suitable solution of (24) can, in fact, be found. Using Eq. 18 one can rewrite (24)

$$\left(\frac{d}{dy} \; + \; y \right) \, \phi_0(y) \; = \; 0 \tag{28}$$

where

$$y \; = \; \sqrt{\frac{m\omega}{\hbar}} \, x \; . \tag{29}$$

Assuming that $\phi_0(y)$ does not vanish anywhere in its domain $]-\infty, +\infty[$ one can write (28)

$$\frac{1}{\phi_0(y)} \, \frac{d}{dy} \phi_0(y) \; = \; \frac{d}{dy} \ln \phi_0(y) \; = \; -\, y \; , \tag{30}$$

the solution of which is

$$\ln\phi_0(y) \;=\; -\frac{1}{2}y^2 \;+\; c_0 \tag{31}$$

for some constant c_0 or

$$\phi_0(y) \;=\; c_0 \exp\left(-\frac{1}{2}y^2 \right) \;. \tag{32}$$

This solution is obviously normalizable. The conventional normalization condition

$$\langle \phi_0 | \phi_0 \rangle_{\Omega_\infty} \;=\; 1 \tag{33}$$

reads

$$\int_{-\infty}^{+\infty} dx\, |\phi_0(x)|^2 \;=\; |c_0|^2 \int_{-\infty}^{+\infty} dx \exp\left(-\frac{m\omega x^2}{\hbar} \right) \;=\; |c_0|^2 \sqrt{\frac{\pi\hbar}{m\omega}} \;. \tag{34}$$

The appropriate ground state is

$$\phi_0(x) \;=\; \left[\frac{m\omega}{\pi\hbar} \right]^{\frac{1}{4}} \exp\left(-\frac{m\omega x^2}{2\hbar} \right) \;. \tag{35}$$

Since the first order differential equation (24) admits only one solution there is only one set of states with energy $E + m\,\hbar\omega$, $m \in \mathbf{Z}$ which properly terminate at some minimum value $E + m_0\,\hbar\omega \geq 0$. We recall that according to (25) the energy value associated with this state is $\frac{1}{2}\hbar\omega$. This state of lowest energy is called the *ground state* of the oscillator. The set of allowed energies of the oscillator according to (23) can then be written as follows

$$E_n \;=\; \hbar\omega\,(\,n \;+\; \frac{1}{2}\,)\,, \quad n \;=\; 0,\, 1,\, 2,\, \ldots \tag{36}$$

It is most remarkable that the energy $\frac{1}{2}\hbar\omega$ of the ground state is larger than the lowest classically allowed energy $E = 0$. The reason is that in the Hamiltonian (2) there are two competing contributions to the energy, the potential energy contribution which for a state $\delta(x)$, i.e., a state confined to the minimum corresponding to the classical state of lowest energy, would yield a vanishing contribution, and the kinetic energy contribution, which for a narrowly localized state yields a large positive value. The ground state (35) assumes a functional form such that both terms together assume a minimum value.

3 Excited States of the Harmonic Oscillator

Having obtained a suitable stationary state with lowest energy, we can now construct the stationary states corresponding to energies (36) above the ground state energy, i.e., we construct the states for $n = 1, 2, \ldots$, the so-called *excited states*. For this purpose we apply the operator $\hat{a}^+$ to the ground state (35) n times. Such states need to be suitably normalized for which purpose we introduce a normalization constant c'_n

$$\phi_n(x) \;=\; c'_n \, \left(\hat{a}^+\right)^n \, \phi_0(x) \,, \quad n \,=\, 0, \, 1, \, 2, \, \ldots \tag{37}$$

These states correspond to the energy eigenvalues (38), i.e., it holds

$$\hat{H} \, \phi_n(x) \;=\; \hbar\omega \, (\, n \,+\, \frac{1}{2} \,) \, \phi_n(x) \,, \quad n \,=\, 0, \, 1, \, 2, \, \ldots \tag{38}$$

We notice that the ground state wave function $\phi_0(x)$ as well as the operators $(\hat{a}^+)^n$ are real. We can, therefore, choose the normalization constants c'_n and the functions $\phi_n(x)$ real as well.

We need to determine now the normalization constants c'_n. To determine these constants we adopt a recursion scheme. For $n = 0$ holds $c'_0 = 1$. We consider then the situation that we have obtained a properly normalized state $\phi_n(x)$. A properly normalized state $\phi_{n+1}(x)$ is then of the form

$$\phi_{n+1}(x) \;=\; \alpha_n \, \hat{a}^+ \, \phi_n(x) \tag{39}$$

for some real constant α_n which is chosen to satisfy

$$\langle \phi_{n+1} | \phi_{n+1} \rangle_{\Omega_\infty} \;=\; \alpha_n^2 \, \langle \hat{a}^+ \phi_n | \hat{a}^+ \phi_n \rangle_{\Omega_\infty} \;=\; 1 \,. \tag{40}$$

Employing the adjoint property yields

$$\alpha_n^2 \, \langle \phi_n | \hat{a}^- \hat{a}^+ \phi_n \rangle_{\Omega_\infty} \;=\; 1 \,. \tag{41}$$

Using (14) together with $\hat{H} \, \phi_n(x) = \hbar\omega(n + \frac{1}{2}) \, \phi_n(x)$ leads to the condition (note that we assumed $\phi_n(x)$ to be normalized)

$$\alpha_n^2 \, (\, n \,+\, 1 \,) \, \langle \phi_n | \hat{a}^- \hat{a}^+ \phi_n \rangle_{\Omega_\infty} \;=\; \alpha_n^2 \, (\, n \,+\, 1 \,) \;=\; 1 \tag{42}$$

From this follows $\alpha_n = 1/\sqrt{n+1}$ and, according to (39),

$$\hat{a}^+ \, \phi_n(x) \;=\; \sqrt{n+1} \, \phi_{n+1}(x) \,. \tag{43}$$

One can conclude then that the stationary states of the oscillator are described by the functions

$$\phi_n(x) \;=\; \frac{1}{\sqrt{n!}} \left(\hat{a}^+\right)^n \, \phi_0(x) \,, \quad n \,=\, 0, 1, 2, \ldots \tag{44}$$

We like to note that these functions according to (27) and the construction (39–44) form an *orthonormal set*, i.e., they obey

$$\langle \phi_n | \phi_{n'} \rangle \;=\; \delta_{n\,n'} \,, \quad n, n' \,=\, 0, 1, 2, \ldots \,. \tag{45}$$

According to (23) holds in analogy to (39)

$$\phi_{n-1}(x) \;=\; \beta_n \, \hat{a}^- \, \phi_n(x) \tag{46}$$

for some suitable constants β_n. Since $\hat{a}^-$ is a real differential operator [see (18)] and since the $\phi_n(x)$ are real functions, β_n must be real as well. To determine β_n we note

$$1 \;=\; \langle \phi_{n-1} | \phi_{n-1} \rangle_{\Omega_\infty} \;=\; \beta_n^2 \, \langle \hat{a}^- \phi_n | \hat{a}^- \phi_n \rangle_{\Omega_\infty} \;=\; \beta_n^2 \, \langle \phi_n | \hat{a}^+ \hat{a}^- \phi_n \rangle_{\Omega_\infty} \, . \tag{47}$$

Equations (15 , 38) yield

$$1 \;=\; \beta_n^2 \, n \, \langle \phi_n | \phi_n \rangle_{\Omega_\infty} \;=\; \beta_n^2 \, n \, . \tag{48}$$

From this follows $\beta_n \;=\; 1/\sqrt{n}$ and, according to (46),

$$\hat{a}^- \, \phi_n(x) \;=\; \sqrt{n} \, \phi_{n-1}(x) \, . \tag{49}$$

Repeated application of this relationship yields

$$\phi_{n-s}(x) \;=\; \sqrt{\frac{(n-s)!}{n!}} \, \left(\hat{a}^- \right)^s \, \phi_n(x) \, . \tag{50}$$

Evaluating the Stationary States

We want to derive now an analytical expression for the stationary state wave functions $\phi_n(x)$ defined through (38). For this purpose we start from expression (44), simplifying the calculation, however, by introducing the variable y defined in (29) and employing the normalization

$$\int_{-\infty}^{+\infty} dy \, \phi_n^2(y) \;=\; 1 \tag{51}$$

This normalization of the wave functions differs from that postulated in (34) by a constant, n–independent factor, namely the square root of the Jacobian dx/dy, i.e., by

$$\sqrt{\left| \frac{dx}{dy} \right|} \;=\; \left[\frac{m\omega}{\hbar} \right]^{\frac{1}{4}} \, . \tag{52}$$

We will later re-introduce this factor to account for the proper normalization (34) rather than (51).

In terms of y the ground state wave function is

$$\phi_0(y) \;=\; \pi^{-\frac{1}{4}} \, e^{-\frac{y^2}{2}} \tag{53}$$

and $\hat{a}^+$ is

$$\hat{a}^+ \;=\; \frac{1}{\sqrt{2}} \left(y \;-\; \frac{d}{dy} \right) \, . \tag{54}$$

The eigenstates of the Hamiltonian are then given by

$$\phi_n(y) \; = \; \frac{1}{\sqrt{2^n \, n! \, \sqrt{\pi}}} \, e^{-\frac{y^2}{2}} \, e^{\frac{y^2}{2}} \left(y \; - \; \frac{d}{dy} \right)^n e^{-\frac{y^2}{2}} \; . \tag{55}$$

Relationship to Hermite Polynomials

We want to demonstrate now that the expression (55) can be expressed in terms of Hermite polynomials $H_n(y)$ which can be given, for example, by the Rodrigues formula. We will demonstrate below the identity

$$H_n(y) \; = \; e^{\frac{y^2}{2}} \left(y \; - \; \frac{d}{dy} \right)^n e^{-\frac{y^2}{2}} \tag{56}$$

such that one can write the stationary state wave functions of the harmonic oscillator

$$\phi_n(y) \; = \; \frac{1}{\sqrt{2^n \, n! \, \sqrt{\pi}}} \, e^{-\frac{y^2}{2}} \, H_n(y) \; . \tag{57}$$

It should be noted that the normalization (27) of the ground state and the definition (56) which includes the factor $1/\sqrt{n!}$ according to Eqs. (39–45) yields a set of normalized states.

To verify the relationship between the definition (56) of the *Hermite* polynomials and the definition given by the Rodrigues formula we need to verify

$$\left(y \; - \; \frac{d}{dy} \right)^n e^{-\frac{y^2}{2}} \; = \; (-1)^n \, e^{\frac{y^2}{2}} \, \frac{d^n}{dy^n} \, e^{-y^2} \; . \tag{58}$$

which implies

$$H_n(y) \; = \; (-1)^n \, e^{y^2} \, \frac{d^n}{dy^n} \, e^{-y^2} \tag{59}$$

and, hence, the *Rodrigues formula*.

We prove (58) by induction noting first that (58) holds for $n = 0, 1$, and showing then that the property also holds for $n+1$ in case it holds for n, i.e.,

$$g(y) \; = \; (-1)^{n+1} \, e^{\frac{y^2}{2}} \, \frac{d^{n+1}}{dy^{n+1}} \, e^{-y^2} \; = \; \left(y \; - \; \frac{d}{dy} \right)^{n+1} e^{-\frac{y^2}{2}} \; . \tag{60}$$

One can factor $g(y)$ and employ (58) as follows

$$\begin{aligned} g(y) \; &= \; -e^{\frac{y^2}{2}} \, \frac{d}{dy} \, e^{-\frac{y^2}{2}} \, (-1)^n \, e^{\frac{y^2}{2}} \, \frac{d^n}{dy^n} \, e^{-y^2} \\ &= \; -e^{\frac{y^2}{2}} \, \frac{d}{dy} \, e^{-\frac{y^2}{2}} \left(y \; - \; \frac{d}{dy} \right)^n e^{-\frac{y^2}{2}} \; . \end{aligned} \tag{61}$$

Denoting $f(y) \; = \; (y \, - \, d/dy)\exp(-y^2/2)$ and employing

$$\frac{d}{dy}\, e^{-\frac{y^2}{2}}\, f(y) \;=\; -y\, e^{-\frac{y^2}{2}}\, f(y) \;+\; e^{-\frac{y^2}{2}}\, \frac{d}{dy}\, f(y) \tag{62}$$

one obtains

$$g(y) \;=\; \left(y \;-\; \frac{d}{dy} \right) f(y) \tag{63}$$

which implies that (60) and, therefore, (58) hold.