Midterm Examination
Physics 386 / Spring 1997
Professor Klaus Schulten

Problem 1: Scattering of a Potential Step (10 pts)

A particle at energy $E > 0$ approaches a potential step from $+\infty$ described by

$$V(x) \;=\; \begin{cases} -V_o & \text{for } x < 0 \\ 0 & \text{for } x \geq \end{cases} \tag{1}$$

where $V_o > 0$. Determine the transmission coefficient and reflection coefficient for scattering at the potential step.

Problem 2: Linear Harmonic Oscillator in the Momentum Representation (10 pts)

The stationary states of the linear harmonic oscillator may be written in the position-space representation as $\phi_n(x)$, or in the momentum representation as $\tilde{\phi}_n(p)$. Using the fact that, in the momentum representation, $\hat{x} = i\hbar\partial_p$, find $\tilde{\phi}_n(p)$. You should not have to do any extensive calculations (i.e. no Fourier transforms).

Problem 3: Particles in a Square Well Potential (10 pts)

A particle of mass m is bound by a potential

$$V(x) \;=\; \begin{cases} 0 & \text{for } |x| > a \\ -V_o & \text{for } |x| \leq a, \text{ where } V_o > 0 \end{cases} \tag{2}$$

For given mass m and width $2a$ how large must V_o be for the existence of three bound states. Hint: Use results derived in class.

Problem 4: Displaced Harmonic Oscillator (20 pts)

Consider a linear harmonic oscillator which experiences an external force that is suddenly switched on at $t = 0$. The Hamiltonian describing the system is

$$\hat{H} \;=\; \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 + F(t), \text{ where} \tag{3}$$

$$F(t) \;=\; \begin{cases} 0 & \text{for } t < 0 \\ F_0 & \text{for } t > 0\,. \end{cases} \tag{4}$$

For $t < 0$ the system is in the ground state,

$$\Psi_0(x,t) = \left[\frac{m\omega}{\pi\hbar}\right]^{\frac{1}{4}} \exp\left(-\frac{m\omega}{2\hbar}x^2\right) \exp\left(-i\frac{\omega t}{2}\right) \,. \tag{5}$$

Since for $t > 0$ the force is time-independent, one can describe the system with stationary states of a harmonic oscillator corresponding to the new potential $V(x) = \frac{1}{2}m\omega^2 x^2 - F_0 x$. These stationary states may be characterized by discrete energy values $E_n^F, (n = 0, 1, \ldots)$ and wave functions

$$\Psi_n^F(x,t) = \psi_n^F(x)\ \exp\left(-i\,\frac{E_n^F}{\hbar}\,t\right)\ , \tag{6}$$

where the F signifies that these are the wave functions for when the force is turned on.

(a) On the same axes, sketch the potential experienced by the particle for (i) $t < 0$ and (ii) $t > 0$.

(b) Show that the stationary states of the displaced oscillator are of the form $\psi_n^F(x) = \phi_n(x - a)$, and that the corresponding energies are of the form $\hbar\omega(n + 1/2) + \Delta E$. What are the values of a and ΔE?

(c) Determine the complete, time-dependent solution for $t > 0$ in the form

$$\psi(x,t) \ = \ \sum_{n=0}^{\infty} c_n(t)\,\psi_n^F(x) \tag{7}$$

where the expansion coefficients $c_n(t)$ are to be determined such that $\psi(x,0) = \psi_0(x,0)$ holds.

(d) Find the value of $\sum_{n=0}^{\infty} |c_n(t)|^2$. By interpreting what the quantities $|c_n(t)|^2$ mean, explain why this result had to hold.

In your derivations you may find useful the following two formulae:

$$\int_{-\infty}^{\infty} dy\ \phi_n(y)\phi_m(y) = \delta_{n,m}$$

and

$$\phi_0(y + a) = \exp(-a^2/4)\sum_{n=0}^{\infty}\frac{(-a)^n}{\sqrt{2^n n!}}\ \phi_n(y)$$

where

$$\phi_n(y) \ = \ \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} e^{-y^2/2} H_n(y)\ .$$