Problem Set 8
Physics 386 / Spring 1997
Professor Klaus Schulten

Problem 1: Expectation Value of Observable

Let $\hat{O}$ be a hermitian operator that represents an observable and let $\{u_n(x),\ n = 1, 2, \ldots\}$ be a complete orthonormal basis of eigenstates of $\hat{O}$ defined through

$$\hat{O}\, u_n(x) \;=\; \lambda_n\, u_n(x)\,. \tag{1}$$

Let $\psi(x)$ describe the state of a quantum system with associated expansion

$$\psi(x) \;=\; \sum_{n=1}^{\infty} c_n\, u_n(x)\,. \tag{2}$$

(a) Argue why

$$\sum_{n=1}^{\infty} \lambda_n\, |c_n|^2 \tag{3}$$

represents the average value to be measured for the observable corresponding to $\hat{O}$. One calls this value the expectation value of $\hat{O}$ in the state $\psi(x)$.

(b) Prove that the expectation value can be written

$$\langle \psi | \hat{O} | \psi \rangle\,, \tag{4}$$

i.e., prove the equivalence of this expression with the expansion (3) above.

Problem 2: Harmonic Oscillator Again

At $t = 0$ a particle is in a state described by the following coherent superposition of harmonic oscilllator stationary states

$$\psi(x, 0) \;=\; \frac{1}{4}\phi_0(x) \;+\; \frac{i}{2}\phi_1(x) \;+\; \frac{\sqrt{11}}{4}\phi_2(x) \tag{5}$$

where $\phi_n(x)$ are the solutions of the time -independent Schrödinger equation

$$\left(-\frac{\hbar^2}{2m}\,\partial_x^2 \;+\; \frac{1}{2} m\omega^2 x^2 \right)\; \phi_n(x) \;=\; \hbar\omega(n + \frac{1}{2})\, \phi_n(x) \tag{6}$$

(a) Determine the wave function at time t.

(b) Determine the probability to measure the energy $\frac{3}{2}\hbar\omega$ at time t.

(c) Argue why the functions $\phi_n(x)$ are orthonormal. (Hint: Consider the energy eigenvalues.)

(d) Evalute the expectation value of the position operator $\hat{x}$ employing expression (4) with $\hat{O} = \hat{x}$. For this purpose express $\hat{x}$ in terms of the creation and annihilation operators defined in class. Remember that, e.g., $a^+\phi_n(x) = \sqrt{n+1}\,\phi_{n+1}(x)$ and that the functions $\phi_n(x)$ are orthonormal.

Problem 3: Electron in a Box

An electron is in the ground state of a box with sides at $x = 0$ and $x = a$. Suddenly, the wall at $x = a$ is moved to to $x = 2a$. Find the probability that the electron will be found (a) in the ground state and (b) in the first excited state of the new box. (c) For which states will the corresponding probability be zero?

This problem set needs to be handed in by Wednesday, April 23, 1997.