Problem 1: Harmonic Oscillator States

Well, plug and chug, that's all there is to it. You'll know if you've done it right or not.

For part d), I'm just looking for a qualitatively accurate sketch. Notice that wave function $\phi_n(x)$ has n nodes, i.e. zeros, just like the wave functions for the square well. You can get a decent sketch by first just drawing the $H_n(y)$ part (don't worry about the constants, just ignore them) and then drawing a Gaussian on top of it, kind of like a wave packet that's modulated by an envelope.

Problem 2: Half-Oscillator

If you stare at this problem long enough, you'll probably convince yourself that the only possible solutions must be the the simple harmonic oscillator solutions. But not all of them will work, only those that equal zero at the origin; in other words, the odd wave functions. The total wave function is then

$$\psi(x) = 0 \quad x \leq 0, \tag{1}$$

$$\psi(x) = \phi_n(x),\ n \text{ odd} \quad x > 0. \tag{2}$$

This wave function evidently satisfies the Schroedinger equation in both regions, and is continous. This problem is similar to problems in electrostatics when we use image charges to solve Poisson's equation. Since you know there's only one possible solution, once you've found any solution, it has to be the right one!

The energy spectrum can be written

$$E_n = \hbar\omega(n + \tfrac{1}{2}) \quad n = 1,\ 3,\ 5, \ldots \tag{3}$$

$$E_m = \hbar\omega(2m + \tfrac{3}{2}) \quad m = 0,\ 1,\ 2, \ldots \tag{4}$$

Problem 3: Annihilation and Creation Operators

(a) To do this proof, remember that the Hamiltonian for the harmonic oscillator can be written

$$\hat{H} = \hbar\omega(a^+a^- + \frac{1}{2}). \tag{5}$$

Since

$$\hat{H}\phi_n(x) = E_n\phi_n(x) = \hbar\omega(n + \frac{1}{2})\phi_n(x), \tag{6}$$

we must have

$$a^+a^-\phi_n(x) = n\phi_n(x). \tag{7}$$

Now, we also know that a^- is a *lowering operator*, which means that

$$a\phi_n(x) = \beta_n\phi_{n-1}(x) \tag{8}$$

for some constant β that depends only on n. We want to find out what β is. We do this by taking advantage of the fact that the $\phi_n(x)$ are normalized:

$$\int_{-\infty}^{\infty} \phi_n^*(x)a^+a^-\phi_n(x)dx = \int_{-\infty}^{\infty} \phi_{n-1}^*\beta_n^*\beta_n\phi_{n-1}(x)dx, \tag{9}$$

$$\int_{-\infty}^{\infty} \phi_n^*(x)\, n\, \phi_n(x) dx = |\beta_n|^2 \int_{-\infty}^{\infty} \phi_{n-1}^*(x)\phi_{n-1}(x)dx, \tag{10}$$

$$|\beta_n|^2 = n, \tag{11}$$

$$\beta_n = \sqrt{n}. \tag{12}$$

You can check the fact that $a^- \phi_n(x) = \beta_n \phi_{n-1}(x)$ implies $\phi_n^*(x)a^+ = \phi_{n-1}^*(x)\beta_n^*$ by taking the complex conjugate. A complete proof would require going into some technical mathematical physics that you can probably wait until grad school to learn about.

Once we have this result, it's easy to see that

$$\left(a^-\right)^s \phi_n(x) = \sqrt{n}\sqrt{n-1}\ldots\sqrt{n-s+1}\phi_{n-s}(x) \tag{13}$$

$$= \sqrt{\frac{n!}{(n-s)!}}\phi_{n-s}(x); \text{ or} \tag{14}$$

$$\phi_{n-s}(x) = \sqrt{\frac{(n-s)!}{n!}}\left(a^-\right)^s \phi_n(x) \tag{15}$$

which is our result.

(b)

There's not much to say about this one, either. I'll show you the algebra I did.

We have

$$\phi_1(x) = A_1 e^{-m\omega x^2/2\hbar} 2\sqrt{\frac{m\omega}{\hbar}}x \quad A_1 \equiv \left(\tfrac{m\omega}{\pi\hbar}\right)^{1/4}\tfrac{1}{\sqrt{2}}, \tag{16}$$

$$\phi_2(x) = A_2 e^{-m\omega x^2/2\hbar}\left(4\frac{m\omega}{\hbar}x^2 - 2\right) \quad A_2 \equiv \left(\tfrac{m\omega}{\pi\hbar}\right)^{1/4}\tfrac{1}{\sqrt{8}}. \tag{17}$$

Using the position-space representation of the raising operator,

$$a^+ f(x) = \left(\sqrt{\frac{m\omega}{2\hbar}}x - \sqrt{\frac{\hbar}{2m\omega}}\frac{d}{dx}\right) f(x), \tag{18}$$

we have

$$a^+\phi_1(x) = \sqrt{\frac{m\omega}{2\hbar}}x 2A_1 e^{-\frac{m\omega x^2}{2\hbar}}\sqrt{\frac{m\omega}{\hbar}}x \tag{19}$$

$$- \sqrt{\frac{\hbar}{2m\omega}}2A_1\sqrt{\frac{m\omega}{\hbar}}\frac{d}{dx}\left(e^{-\frac{m\omega x^2}{2\hbar}}x\right) \tag{20}$$

$$= \frac{m\omega}{\hbar}\sqrt{2}x^2 A_1 e^{-(\ldots)} - \sqrt{2}A_1\left(e^{-(\ldots)} - \frac{m\omega}{\hbar}x^2 e^{-(\ldots)}\right) \tag{21}$$

$$= A_1 e^{-(\ldots)}\sqrt{2}\left(\frac{2m\omega}{\hbar}x^2 - 1\right) \tag{22}$$

$$
\begin{aligned}
&= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega x^2}{2\hbar}} \frac{1}{2}\left[4\left(\sqrt{\frac{m\omega}{\hbar}}x\right)^2 - 2\right] && (23)\\
&= \sqrt{2}\left\{\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{8}} e^{-\frac{m\omega x^2}{2\hbar}} H_2\left(\sqrt{\frac{m\omega}{\hbar}}x\right)\right\} && (24)\\
&= \sqrt{2}\phi_2(x). && (25)
\end{aligned}
$$

(c)

To prove this commutator property, we'll start by proving the general relation for $[A, BC]$ where A, B, and C are operators. It's easiest to work backwards:

$$
\begin{aligned}
B[A,C] + [A,B]C &= B[AC - CA] + [AB - BA]C && (26)\\
&= BAC - BCA + ABC - BAC && (27)\\
&= ABC - BCA && (28)\\
&= [A, BC]\,, \text{ so that} && (29)\\
[A, BC] &= B[A,C] + [A,B]C. && (30)
\end{aligned}
$$

Applying this formula to our case, we have

$$
[A, B^n] = [A, B^{n-1} \cdot B] = B^{n-1}[A,B] + [A, B^{n-1}]B. \tag{31}
$$

We do mathematical induction when we want to prove that a statement is true for all integers n. Mathematical induction involves three steps:

- Prove a base case, e.g. for $n = 1$.
- Assume the statement you are trying to prove is true for all $k \le n$. This is your *inductive hypothesis.*
- Using the inductive hypothesis, prove the case for $n = k + 1$.

We proved the base case for $n = 1$ in class: $[a^-, a^+] = 1$. Our inductive hypothesis is that

$$
[a^-, (a^+)^k] = k\ (a^+)^{k-1}\,,\ n = 1,\ 2,\ \ldots k. \tag{32}
$$

Now we want to prove the case $k = 1$, using our inductive hypothesis and any other facts we know, such as the commutator property we proved above. We have

$$
\begin{aligned}
[a^-, (a^+)^{k+1}] &= (a^+)^k [a^-, a^+] + [a^-, (a^+)^k]a^+ && (33)\\
&= (a^+)^k + [a^-, (a^+)^k]a^+ && (34)\\
&= (a^+)^k + k\ (a^+)^{k-1} a^+ && (35)\\
&= (k+1)(a^+)^k. && (36)
\end{aligned}
$$

We conclude that Eq. 32 is true for *all* integers k.