Problem Set 7
Physics 386 / Spring 1997
Professor Klaus Schulten

Problem 1: Harmonic Oscillator States

In class we have derived the stationary states of the harmonic oscillator using creation and annihilation operators. In this problem demonstrate that the wave functions

$$\phi_n(x) \;=\; \frac{1}{\sqrt{2^n\, n!}} \left[\frac{m\omega}{\pi\hbar}\right]^{\frac{1}{4}} e^{-\frac{m\omega x^2}{2\hbar}}\, H_n(\sqrt{\frac{m\omega}{\hbar}}x)\;. \tag{1}$$

for $n \;=\; 0, 1$ are indeed solutions of the Schrödinger equation

$$\left(-\frac{\hbar^2}{2m}\partial_x^2 \;+\; \frac{1}{2}m\omega^2 x^2 \right) \phi_n(x) \;=\; \hbar\omega(\, n \;+\; \frac{1}{2})\, \phi_n(x). \tag{2}$$

The functions $H_n(y)$ in (1) represent the Hermite polynomials which are for $n \;=\; 0, 1, 2, 3$

$$H_0(y) \;=\; 1, \quad H_1(y) \;=\; 2y, \quad H_2(y) \;=\; 4y^2 \;-\; 2, \quad H_3(y) \;=\; 8y^3 \;-\; 12y \tag{3}$$

(a) Demonstrate (2) for $n \;=\; 0$.

(b) Demonstrate (2) for $n \;=\; 1$.

(c) Demonstrate the orthogonality of $\phi_0(x)$ and $\phi_1(x)$, i.e.,

$$\langle \phi_0 | \phi_1 \rangle \;=\; \int_{-\infty}^{+\infty} dx\, \phi_0\, \phi_1 \;=\; 0\;. \tag{4}$$

(d) Plot $\phi_n(x)$ for $n \;=\; 0, 1, 2, 3$.

Problem 2 : Half-Oscillator

What are the energy levels of a particle of mass m moving in the one-dimensional potential well defined by

$$V(x) \;=\; \begin{cases} \infty & \text{for } x \;<\; 0 \\ \frac{1}{2}m\omega^2 x^2 & \text{for } x \;\geq\; 0 \end{cases}\;. \tag{5}$$

(Hint: You don't need a lengthy calculation.)

Problem 3 : Annihilation and Creation Operators

(a) Prove the property

$$\phi_{n-s}(x) \;=\; \sqrt{\frac{(n-s)!}{n!}}\; \left(\hat{a}^{-}\right)^{s}\, \phi_n(x)\,. \tag{6}$$

(b) Demonstrate, using (1, 3), the property

$$\hat{a}^{+}\, \phi_n(x) \;=\; \sqrt{n+1}\,\phi_{n+1}(x) \tag{7}$$

for $n \,=\, 1$ employing the expression

$$\hat{a}^{+} \;=\; \sqrt{\frac{m\omega}{2\hbar}}\, x \;-\; \sqrt{\frac{\hbar}{2m\omega}}\, \frac{d}{dx}\,. \tag{8}$$

(c) Demonstrate the commutator property

$$[A,\, B^n] \;=\; B^{n-1}\,[A, B] \;+\; [A,\, B^{n-1}]\, B \tag{9}$$

and then by induction

$$[\hat{a}^{-},\, (\hat{a}^{+})^{n}] \;=\; n\, \left(\hat{a}^{+}\right)^{n-1}\,. \tag{10}$$

This problem set needs to be handed in by Wednesday, April 9, 1997.