Problem 1: Barrier Penetration

We have a square barrier of width 10^{-7} cm and height $V_o = 6\ eV$. Electrons with energy 5 eV are incident on the barrier. You are asked to calculate the transmission coefficient.

Notation: By the transmission coefficient I will always mean the ratio of the incident flux to the transmitted flux, and denote by $|T|^2$. I will use T to denote the complex *amplitude* of the transmitted wave. $|R|^2$ and R denote the reflection coefficient and amplitude of the reflected wave, respectively, and the incoming wave is assumed to have unit amplitude. This is the notation used by Goswami (pg. 76 ff) and is not the same as the notation used in class; I choose this notation because that's what most students used in the homework.

A derivation of the transmission coefficient is given below; the formula itself was given in class and can be found in Goswami as equation (4.13):

$$|T|^2 = \frac{(2k\alpha)^2}{(k^2+\alpha^2)^2\, sinh^2 2\alpha a + (2k\alpha)^2}, \tag{1}$$

where

$$\alpha^2 = \frac{2m(V_0 - E)}{\hbar^2}, \tag{2}$$

$$k^2 = \frac{2mE}{\hbar^2}. \tag{3}$$

Most students began with this formula, plugged in the corresponding parameters in the problem, and obtained the **wrong** answer. Some students used the wrong value of a by factor of 2; remember that the width of the barrier in this dervation is not a, but $2a$. Many students, though, simply got the numbers and/or units mixed up.

In my experience, mistakes like these happen a lot less often when one uses a set of units that are appropriate to the problem. If you use MKS units to work your quantum problems, you will have lots of exponents floating around that are easy to lose track of. Here are some tips:

- Measure energy in eV or MeV.
- Convert all masses to eV/c^2, e.g. $m_e c^2 = 511 \times 10^6 eV$.
- Turn your $\hbar$'s into $\hbar c = 1973\ eV\mathring{A}$.
- If you still want to use some other system of units, double check that you've used the *same* system units in each expression.
- Use dimensional analysis to see if an expression makes sense.

Answers:

$$\begin{aligned} k &= 1.1457\text{Å}^{-1} && (4)\\ \alpha &= 0.5124\text{Å}^{-1} && (5)\\ |T|^2 &= 7.87 \times 10^{-5} && (6) \end{aligned}$$

Derivation of transmission coefficient

In region I, II, and III, corresponding to the left, middle and right regions of the problem, the wave function has the form

$$\begin{aligned} \psi_I(x) &= e^{ikx} + Re^{-ikx}; && (7)\\ \psi_{II}(x) &= A\ sinh(\alpha x) + B\ cosh(\alpha x) && (8)\\ \psi_{III}(x) &= Te^{ikx} && (9) \end{aligned}$$

Enforcing continuity of the wave function at the two boundaries gives us

$$\begin{aligned} e^{-ika} + Re^{ika} &= -A\ sinh\beta + B\ cosh\beta && (10)\\ Te^{ika} &= A\ sinh\beta + B\ cosh\beta, && (11) \end{aligned}$$

where $\beta = \alpha a$, and enforcing the continuity of the first derivative gives us

$$\begin{aligned} ik\left(e^{-ika} - Re^{ika}\right) &= \alpha\left(A\ cosh\beta - B\ sinh\beta\right), && (12)\\ T\ ik\ e^{ika} &= \alpha\left(A\ cosh\beta + B\ sinh\beta\right). && (13) \end{aligned}$$

Add and subtract Eq.(10) and (11) to get

$$\begin{aligned} (T+R)\,e^{ika} + e^{-ika} &= 2B\ cosh\beta, && (14)\\ (T-R)\,e^{ika} - e^{-ika} &= 2A\ sinh\beta. && (15) \end{aligned}$$

Do the same with Eq.(12) and (13) to get

$$\begin{aligned} ik\left[(T-R)\,e^{ika} + e^{-ika}\right] &= 2\alpha A\ cosh\beta, && (16)\\ ik\left[(T+R)\,e^{ika} - e^{-ika}\right] &= 2\alpha B\ sinh\beta. && (17) \end{aligned}$$

Now use Eq.(14) and (15) to solve for A and B. When we put these into Eq.(16) and (17), we have eliminated A and B and are left with

$$\begin{aligned} (T-R)\,e^{ika} + e^{-ika} &= \frac{\alpha}{ik}\frac{cosh\beta}{sinh\beta}\left[(T-R)\,e^{ika} - e^{-ika}\right], && (18)\\ (T+R)\,e^{ika} - e^{-ika} &= \frac{\alpha}{ik}\frac{sinh\beta}{cosh\beta}\left[(T+R)\,e^{ika} + e^{-ika}\right], && (19)\\ & && (20) \end{aligned}$$

which can be solved for $(T-R)$ and $(T+R)$:

$$T - R = e^{-ika}\frac{-\frac{\alpha}{ik}\frac{cosh\beta}{sinh\beta}e^{-ika} - e^{-ika}}{1-\frac{\alpha}{ik}\frac{cosh\beta}{sinh\beta}} \tag{21}$$

$$= e^{-2ika}\frac{\alpha cosh\beta + ik\ sinh\beta}{\alpha cosh\beta - ik\ sinh\beta}, \tag{22}$$

$$T + R = e^{-2ika}\frac{ik\ cosh\beta + \alpha\ sinh\beta}{ik\ cosh\beta - \alpha\ sinh\beta}. \tag{23}$$

Finally, we add these two equations to get T. In the numerator, we use the identity $cosh^2\beta - sinh^2 beta = 1$ to cancel some terms. In the denominator, we use $cosh^2\beta + sinh^2\beta = cosh2\beta$ and $cosh\beta\ sinh\beta = 1/2\ sinh2\beta$ to obtain

$$T = e^{-2ika}\frac{2ik\alpha}{2ik\alpha\ cosh2\beta + (k^2-\alpha^2)\,sinh2\beta} \tag{24}$$

$$= e^{-2ika}\frac{2k\alpha}{2k\alpha\ cosh2\beta - i\,(k^2-\alpha^2)\,sinh2\beta}, \tag{25}$$

$$|T|^2 = \frac{4k^2\alpha^2}{4k^2\alpha^2 cosh^2 2\beta + (k^2-\alpha^2)^2\,sinh^2 2\beta} \tag{26}$$

$$= \frac{4k^2\alpha^2}{4k^2\alpha^2 + (k^2+\alpha^2)^2\,sinh^2 2\beta}. \tag{27}$$

Problem 2: Transmission at a step

We have a plane wave incident from the left on a potential step of height V_o that begins at the origin. We expect that there will be a transmitted wave and a reflected wave. The wave function in the two regions is given by

$$\psi_1(x) = e^{ikx} + Re^{-ikx} \quad x < 0 \tag{28}$$

$$\psi_2(x) = Te^{i\alpha x} \quad x > 0 \tag{29}$$

where

$$k = \frac{\sqrt{2mE}}{\hbar}, \tag{30}$$

$$\alpha = \frac{\sqrt{2m\,(E-V_o)}}{\hbar}. \tag{31}$$

Notice that α is either purely real or purely imaginary.

We apply the conditions of continuity of the wave function and its first derivative at the origin to obtain

$$1 + R = T, \tag{32}$$

$$ik(1-R) = i\alpha T, \tag{33}$$

$$T = \frac{2k}{k+\alpha}, \tag{34}$$

$$|T|^2 = \frac{4k^2}{|k+l|^2}. \tag{35}$$

Whether α is real or not, the transmitted flux is given by

$$\begin{aligned} j_T &= \frac{\hbar}{2mi}|T|^2(i\alpha + i\alpha^*) && (36) \\ &= \frac{\hbar}{m}|T|^2 \; Re(\alpha). && (37) \end{aligned}$$

Since the incident flux is $\hbar k/m$, the transmission coefficient is equal to

$$|T|^2 = Re(\alpha)/k. \tag{38}$$

For $E < V_o$, α is purely imaginary, so $|T|^2 = 0$. For $E > V_o$, α is real and can be treated like any old number. In that case

$$\begin{aligned} T &= \frac{4k^2}{(k+\alpha)^2} \times \frac{\alpha}{k} && (39) \\ &= \frac{4k\alpha}{k^2+\alpha^2+2k\alpha} && (40) \\ &= \frac{4\sqrt{E(E-1)}}{2E+2\sqrt{E(E-1)}-1}. && (41) \end{aligned}$$

Problem 3: Transmission for a potential step II

Just take the solution we already have for a wave moving from the left and change $x \to -x$. The amplitudes of the respective waves are unchanged, so we have

$$\psi(x) = e^{-ikx} + \frac{k-\alpha}{k+\alpha}e^{ikx} \quad x > 0 \tag{42}$$

$$\psi(x) = \frac{2k}{k+\alpha}e^{-i\alpha x} \quad x < 0 \tag{43}$$

Note that there's no point in considering $E < V_o$ since then there would be no incident wave.

Problem 4: Scattering off a rectangular barrier

You could work this problem out the same way we did for Problem 1 above, but the suave way to do it is to realize that the equations we have to solve are exactly the same, except that now α is imaginary instead of real, since $E > V_o$. Define $iq \equiv \alpha$, so that q is real. Notice that, formally,

$$cosh \; ix = \frac{e^{ix} + e^{-ix}}{2} \tag{44}$$

$$
\begin{aligned}
&= cos\ x && (45)\\
sinh\ ix &= \frac{e^{ix} - e^{-ix}}{2} && (46)\\
&= i\ sin\ x && (47)
\end{aligned}
$$

Redefining $\beta = qa$, we immediately see that, in this case,

$$
\begin{aligned}
T &= e^{-2ika} \frac{2iqk}{2iqk\ cos 2\beta + (k^2+q^2) sin 2\beta}, && (48)\\
|T|^2 &= \frac{4q^2k^2}{4q^2k^2 cos^2 2\beta + (k^2+q^2)^2 sin^2 2\beta} && (49)\\
&= \frac{4q^2k^2}{4q^2k^2 + (k^2-q^2)^2 sin^2 2\beta} && (50)\\
&= \frac{4q^2k^2}{4q^2k^2 + (2mV_o/\hbar^2)^2 sin^2 2\beta.} && (51)
\end{aligned}
$$

Perfect transmission, or resonances occur when $\beta = n\pi/2$. But $\lambda = 2\pi/q$ is the de Broglie wavelength of the electron in the region above the barrier, so these resonances occur when $d = n\lambda/2$, where d is the width of the barrier.