Problem 1: Quantum and Classical particles

You were asked to determine how the size of a wave packet changes with time for two different particles: an electron with kinetic energy $KE = 100\ eV$ travelling a distance of 1 km, and an arrow of mass 200 g travelling for ten seconds. The initial size of the electron wave packet is 10^{-6} m and that of the arrow is 1 cm.

We solve this problem by applying the formula given in class for the probability distribution $P(x,t)$ of a Gaussian wave packet as a function of time. Ignoring the normalization factor, the distribution is proportional to a Gaussian that is drifting and broadening with time:

$$P(x,t) \;\propto\; exp\left[-\frac{\left(x - \frac{p_o}{m}t\right)^2}{\sigma^2}\right], \tag{1}$$

where

$$\sigma^2 \;=\; \delta^2\left(1 + \frac{\hbar^2 t^2}{m^2\delta^4}\right) \tag{2}$$

Just for definiteness, we defined the size of the wave packet to be the uncertainty in the position. For a distribution of type 1, this is equal to

$$\Delta x^2 \;=\; \left\langle x^2\right\rangle - \left\langle x\right\rangle^2 \tag{3}$$

$$\;=\; \frac{1}{2}\sigma^2 \tag{4}$$

Hence, if the size is d, then we can find what δ is by setting $t = 0$ and solving:

$$d^2 \;=\; \frac{1}{2}\delta^2, \tag{5}$$

$$d \;=\; \frac{\delta}{\sqrt{2}}. \tag{6}$$

Once we have δ, we can get the size at any future time by the relation

$$d^2 = \frac{1}{2}\delta^2\left(1 + \frac{\hbar^2 t^2}{m^2\delta^4}\right). \tag{7}$$

No matter how you calculate the size, the size of the wave packet grows by the square root of the factor in parentheses above.

For the electron, we can calculate the time of travel by setting the kinetic energy equal to $1/2mv^2$, solving for v, and dividing by the distance. For the arrow we already have the time. In the case of the arrow the spreading is so small that we can use the approximation $(1+x)^{1/2} \approx 1 + 1/2x$ to get the size at the end of the trip.

Answers: Electron: $d \;=\; .976$ cm

Arrow: $d \;=\; (1 + 3.5 \times 10^{-58})$ cm

Problem 2: Expectation values in momentum space

We have the wave function

$$\phi(x) = \sqrt{\frac{2\hbar^3}{\pi\delta^3}}\; e^{ik_o x}\frac{1}{x^2+\hbar^2/\delta^2} \tag{8}$$

a) Checking the normalization involves doing the same integral we did on the last assignment,

$$\int_{-\infty}^{\infty} dx \frac{1}{(x^2+a^2)^2} = \frac{\pi}{2a^3}, \tag{9}$$

with the substition $a = \hbar^2/\delta^2$. We can ignore the complex exponential for now since it doesn't affect the normalization (remember that we take the modulus squared, not just the square).

b) Computing the momentum representation of the wave function essentially means performing the Fourier transform (see Goswami, p. 53)

$$\tilde{\phi}(p) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} dx\; phi(x)\; e^{-ipx/\hbar} \tag{10}$$

$$= \frac{N}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} dx \frac{1}{x^2+a^2} e^{i(k_o - p/\hbar)x} \tag{11}$$

where N is the normalization constant of $\phi(x)$ and a is as above.

One way to proceed is to make the substition $q = k_o - p/\hbar$ and write the exponential as $cos(qx)+i\;sin(qx)$. The $sin(qx)$ part drops out because of parity, leaving just the $cos(qx)$ term. Conveniently enough we have a formula for this integral. Being careful to put in absolute value signs on q, we have the result

$$\tilde{\phi}(p) = \frac{1}{\sqrt{\delta}}\; exp\left[-\frac{\hbar}{\delta}\left|\frac{p}{\hbar} - k_o\right|\right]. \tag{12}$$

c) Because we used a symmetric definition of the Fourier transform, both representations of the wave function will have the same normalization. The tricky part of doing this integral is keeping track of where the argument inside the absolute value signs is positive and where it is negative. Probably the easiest thing to do here and on part d) is to substitute $y = p/\hbar - k_o$. For part c), you are left with

$$\int_{-\infty}^{\infty} |\tilde{\phi}(p)|^2 = \hbar/\delta \int_{-\infty}^{\infty} dy\; exp\left(-\frac{2\hbar}{\delta}|y|\right) \tag{13}$$

$$= 2\hbar/\delta \int_{0}^{\infty} dy\; exp\left(-\frac{2\hbar}{\delta}y\right) \tag{14}$$

$$= 1. \tag{15}$$

d) Making the same substition as above, we get

$$\begin{aligned}
\langle p \rangle &= 1/\delta \int_{-\infty}^{\infty} \hbar\ dy\ \hbar(y+k_o)\ exp\left[-\frac{2\hbar}{\delta}|y|\right] && (16)\\
&= \frac{\hbar^2 k_o}{\delta} \int_{-\infty}^{\infty} dy\ e^{-2\hbar|y|/\delta} && (17)\\
&= \frac{2\hbar^2 k_o}{\delta}\ \frac{\delta}{2\hbar} && (18)\\
&= \hbar k_o. && (19)
\end{aligned}$$

Problem 3: Wave Packet

The problem was to derive an expression for the complete time dependent wave function of a Gaussian wave packet. Getting the answer into the form requested is tricky; if you weren't able to do it, don't feel bad, you didn't miss much. All of you, however, should have at least been able to do the Gaussian integral.

The expression you were given could be expanded into a quadratic polynomial. Some of the coefficients were imaginary, but that didn't matter in this case. For the record, here's the formula for how to do the most general kind of Gaussian integral:

$$\int_{-\infty}^{\infty} dk\ exp\left(-a\ k^2 + bk + c\right) = e^c\ \sqrt{\pi/a}\ e^{b^2/4a} \tag{20}$$