Problem 1: Fourier Transform

Fourier transforms relate the representation of a function in position space (the way we normally think of functions) its representation in momentum space. On page 28 of Goswami one possible definition of the Fourier transform is given:

$$\psi(x, x_o) = \int_{-\infty}^{\infty} dk\ a(k, k_o) exp[ik(x - x_o)] \tag{1}$$

Here ψ is the position-space representation and a is the momentum space representation. With this definition, the inverse Fourier transform *must* be taken to be

$$a(k, k_o) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx\ \psi(x, x_o) exp[-ix(k - k_o)] \tag{2}$$

Alternatively, one can define the Fourier transform (p. 53 of Goswami) so that both the forward and inverse transform have a factor of $(2\pi)^{1/2}$; this is nice because then both representations have the same normalization. As long as the product of the two prefactors is $1/2\pi$ and the sign of the exponential is opposite in the two cases, you have a valid definition of the Fourier transform. All that matters is that you can take a function, apply the forward and inverse transform, and get back the same function. *Just be sure* to always state your definitions.

The *constants* x_o and k_o tell where the functions are centered in position and momentum space, respectively. In most cases it's simpler just to set $k_o = 0$, since having a nonzero k_o just results in an overall shift of the momementum function.

For this problem $\psi(x)$ is centered at 0 (there isn't any x_o parameter), and there's no reason to introduce a nonzero k_o, so we take both of them to be zero in the above two equations.

Answer: $\frac{1}{2\pi}\left(\frac{sin(k/2)}{k/2}\right)^2$.

Note that this function is even, just like $\psi(x)$.

Problem 2: Expectation Values in Momentum Space

We have

$$\begin{aligned}
\langle k \rangle &= \frac{1}{\sqrt{\pi}\delta k} \int_{-\infty}^{\infty} dk\ exp\left(-\frac{(k - k_o)^2}{\delta k^2}\right) &(3)\\
&= \frac{1}{\sqrt{\pi}\delta k} \int_{-\infty}^{\infty} dy\ exp\left(-\frac{y^2}{\delta y^2}\right)(y + k_o) &(4)\\
&= k_o. &(5)
\end{aligned}$$

For $\langle k^2 \rangle$, after making the substition $y = k - k_o$, we have three integrals to do:

$$\int_{-\infty}^{\infty} dy\ exp\left(-\frac{y^2}{\delta k^2}\right) y^2 + \int_{-\infty}^{\infty} dy\ exp\left(-\frac{y^2}{\delta k^2}\right) k_o^2 + \int_{-\infty}^{\infty} dy\ exp\left(-\frac{y^2}{\delta k^2}\right) 2\ k_o y \tag{6}$$

The third integral is zero by parity. The second is a simple Gaussian integral since k_o^2 can be factored out. The first integral can be done without even integration by parts by use of the following trick:

$$\int dy\ exp(-a\ y^2) = \sqrt{\pi/a}. \tag{7}$$

$$-\frac{d}{da}\int dy\ exp(-a\ y^2) = \int dy\ y^2\ exp(-a\ y^2) \tag{8}$$

$$= \frac{1}{2}\sqrt{\pi} a^{3/2} \tag{9}$$

Substituting $a = 1/\delta k^2$, we find

$$\langle k^2 \rangle = \frac{\delta k^2}{2} + k_o^2, \tag{10}$$

$$\Delta k = \sqrt{\frac{\delta k^2}{2} + k_o^2 - k_o^2} \tag{11}$$

$$= \frac{\delta k}{\sqrt{2}} \tag{12}$$

from which the final result follows immediately.

Problem 3: Ground State of the Harmonic Oscillator

We're asked to use the Heisenberg uncertainty relationship to estimate the ground state energy of the simple harmonic oscillator.

The state of a harmonic oscillator (a pendulum, if you want) is defined by two coordinates, the displacement x and the momentum p. In terms of these coordinates, the energy of the oscillator system is

$$E = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2. \tag{13}$$

Now, what if the harmonic oscillator is a quantum mechanical system? Then the coordinates x and p don't have well-defined values at any instant in time. However, we *can* find expectation values of the momentum and position, so that

$$\langle E \rangle = \frac{\langle p^2 \rangle}{2m} + \frac{1}{2}m\omega^2 \langle x^2 \rangle, \tag{14}$$

the expectation value of the energy E, is a well-defined quantity that we could measure experimentally.

The uncertainty relationship tells us that we will always observe a *spread* in the values of x and p. With the definitions

$$\begin{aligned} \Delta x &= \sqrt{\langle x^2 \rangle - \langle x \rangle^2}, & (15) \\ \Delta p &= \sqrt{\langle p^2 \rangle - \langle p \rangle^2}, & (16) \end{aligned}$$

we can state mathematically that

$$\Delta x \; \Delta p \; \geq \; \hbar. \tag{17}$$

Since $\langle x \rangle = 0$ and $\langle p \rangle = 0$, we have $\Delta x^2 = \langle x^2 \rangle$ and $\Delta p^2 = \langle p^2 \rangle$. By solving for Δx in terms of Δp, or vice versa, we can minimize the expectation value of the energy; the result will presumably correspond to the ground state energy.

The actual value we obtain depends, of course, on what the uncertainty in x and p actually is. Is it $\hbar$? Or $\hbar/2$? Or some other multiple of $\hbar$? Unless you happen to know the form of the wave function for the quantum mechanical system whose energy you are trying to minimize, there's no really good way to say. Therefore, except in rare circumstances, the uncertainty relationships only tell you *order of magnitude estimates* for the energy. They are very useful as a quick rule of thumb to gain qualitative insight into a problem, but not usually as a means to obtain the quantitatively correct answer. For that, it is necessary to solve the Schroedinger equation.

For the hydrogen atom, for example, we obtained the correct ground state energy by using the uncertaintly relationship $\Delta p \Delta x = \hbar$. But we never really justfied why $\hbar$ was the correct choice, except that it gave the right answer! Even so, the fact that we can get remarkably close this way tells us that we have captured most of the essential physics of the problem.

For the simple harmonic oscillator, it just so happens (we will prove this later) that the ground state wave function is Gaussian. Therefore (p. 32 of Goswami) the uncertainty relationship is $\Delta p \Delta x = \hbar/2$. If you use this value, you will get the correct ground state energy, $\hbar\omega/2$.

Problem 4: Life Time Broadening

We have $\Delta t \; = \; 10^{-9}$ sec. If we take

$$\Delta E \Delta t = \hbar, \tag{18}$$

then

$$\Delta E \; = \; \frac{\hbar}{\Delta t} \; = \; \frac{6.6 \times 10^{-16} \; eV - sec}{10^{-9} sec} \; = \; 6.6 \times 10^{-7} \; eV. \tag{19}$$