Problem 1: Avoided Crossing

(a) The eigenvalues of the given hamiltonian are found by setting the determinant

$$\begin{vmatrix} \epsilon p - \lambda & v \\ v & -\epsilon p - \lambda \end{vmatrix} = \lambda^2 - (\epsilon p)^2 - v^2 = 0 \ . \qquad (1)$$

The eigenvalues are then

$$\begin{aligned} \lambda &= \pm\sqrt{(\epsilon p)^2 + v^2} \\ &= \pm\frac{1}{v}\sqrt{\left(\frac{\epsilon p}{v}\right)^2 + 1} \ . \end{aligned}$$

We can now plot the energy eigenvalues vs. p. The easiest way to see the form of the graph is to sketch in the asymptotes $E = \epsilon p$ and $E = -\epsilon p$; the positive energy curve crosses the vertical axis at $+v$ and goes asymptotically to $\pm\epsilon p$ while the negative curve does the mirror image.

(b) Let $E_{1,2}$ be the energy eigenvalues corresponding to $+$ and $-$, respectively. Using the top row of the hamiltonian, we get

$$\begin{aligned} \epsilon p \alpha_{1,2} + v\beta_{1,2} &= E_{1,2}\alpha_{1,2} \ , \\ \beta_{1,2} &= \frac{E_{1,2} - \epsilon p}{v}\alpha_{1,2} \ ; \\ \begin{pmatrix} \alpha_{1,2} \\ \beta_{1,2} \end{pmatrix} &= \left(1 + \left(\frac{E_{1,2} - \epsilon p}{v}\right)^2\right)^{-1/2} \begin{pmatrix} 1 \\ \frac{1}{v}\left(E_{1,2} - \epsilon p\right) \end{pmatrix} \ . \end{aligned}$$

To plot the eigenvectors, ignore the normalization factor out front for now and focus on the ratio between the two components. For the eigenvector corresponding to the positive eigenvalue E_1, we see that the coefficients are always positive; moreover, β_1 is large compared to α_1 for very negative p but small for very large and positive p. In order to keep the overall normalization, this means that α_1 and β_1 must go to zero from the positive side as p goes to negative and positive infinity, respectively, while the other coefficient goes to 1. They meet at $p = 0$ at the value $1/\sqrt{2}$. For the eigenvector corresponding to the negative eigenvalue E_2, a similar analysis shows that α_2 goes to $+1$ and β_2 goes to -0 as $p \to -\infty$ (β_2 is always negative); as $p \to +\infty$, α_2 and β_2 go to $+0$ and -1, respectively.

Why do we call this an 'avoided crossing'? Notice that if $v = 0$, then as we vary the parameter p the energy eigenvalues would cross at $p = 0$. However, if v is only slightly different from zero, then the eigenvalues do not cross.

Problem 2: Four State Quantum System

(a) By permuting basis states 2, 3, and 4 cyclically, we can put the hamiltonian into the form

$$\hat{H} \to \begin{pmatrix} a & -b & & \\ -b & -a & & \\ & & c & id \\ & & -id & -c \end{pmatrix} \ . \qquad (2)$$

(with 0 in the off-diagonal blocks)

The eigenvalues are $E_{1,2} = \sqrt{a^2 + b^2}$ and $E_{3,4} = \pm\sqrt{c^2 + d^2}$. The eigenvectors $|\tilde{1}\rangle$ and $|\tilde{2}\rangle$, corresponding to $E_{1,2}$, are linear combinations of the kets $|1\rangle$ and $|4\rangle$ of the original basis. Similarly the eigenvectors $|\tilde{3}\rangle$ and $|\tilde{4}\rangle$ will be linear combinations of kets $|2\rangle$ and $|3\rangle$ of the original basis.

Working as we did in Problem 1, we find that the coefficients of the eigenstates are

$$\begin{aligned} \begin{pmatrix} \langle 1|\tilde{1}, \tilde{2}\rangle \\ \langle 4|\tilde{1}, \tilde{2}\rangle \end{pmatrix} &= \begin{pmatrix} \alpha_{1,2} \\ \beta_{1,2} \end{pmatrix} = \left(1 + \left(\frac{a - E_{1,2}}{b}\right)^2\right)^{-1/2} \begin{pmatrix} 1 \\ \frac{1}{b}\left(a - E_{1,2}\right) \end{pmatrix} \ ; \\ \begin{pmatrix} \langle 2|\tilde{3}, \tilde{4}\rangle \\ \langle 3|\tilde{3}, \tilde{4}\rangle \end{pmatrix} &= \begin{pmatrix} \alpha_{3,4} \\ \beta_{3,4} \end{pmatrix} = \left(1 + \left(\frac{c - E_{3,4}}{d}\right)^2\right)^{-1/2} \begin{pmatrix} 1 \\ \frac{i}{d}\left(c - E_{3,4}\right) \end{pmatrix} \ . \end{aligned}$$

(b) If the system is originally in the state $|\psi(0)\rangle = |2\rangle$, then it can be represented as a linear combination of eigenstates $|\tilde{3}\rangle$ and $|\tilde{4}\rangle$, since the other two eigenstates don't have any $|2\rangle$ components. We have to set up and solve the system of equations given by

$$\begin{pmatrix}1\\0\end{pmatrix} = c_1\left(1+\left(\frac{c-E_3}{d}\right)^2\right)^{-1/2}\begin{pmatrix}1\\ \frac{i}{d}(c-E_3)\end{pmatrix} + c_2\left(1+\left(\frac{c-E_4}{d}\right)^2\right)^{-1/2}\begin{pmatrix}1\\ \frac{i}{d}(c-E_4)\end{pmatrix}. \tag{3}$$

If we define

$$\begin{aligned} d_1 &\equiv \frac{c_1}{\sqrt{1+\left(\frac{c-E_3}{d}\right)^2}} \text{ and} \\ d_2 &\equiv \frac{c_2}{\sqrt{1+\left(\frac{c-E_4}{d}\right)^2}}, \end{aligned}$$

then we obtain simple equations for d_1 and d_2, which we can solve and then back-substitute to get

$$\begin{aligned} d_1 &= \frac{c-E_4}{E_3-E_4}, \\ d_2 &= 1-d_1 = \frac{E_3-c}{E_3-E_4}; \\ c_1 &= \sqrt{1+\left(\frac{c-E_3}{d}\right)^2}\,\frac{c-E_4}{E_3-E_4}, \\ c_2 &= \sqrt{1+\left(\frac{c-E_4}{d}\right)^2}\,\frac{E_3-c}{E_3-E_4}. \end{aligned}$$

Combining these results, and using our knowlege of how eigenstates of the hamiltonian evolve in time, we obtain the state vector at time time t:

$$\begin{aligned} |\psi(t)\rangle &= e^{-iE_3t/\hbar}\frac{c-E_4}{E_3-E_4}\begin{pmatrix}1\\ \frac{i}{d}(c-E_3)\end{pmatrix} + e^{-iE_4t/\hbar}\frac{E_3-c}{E_3-E_4}\begin{pmatrix}1\\ \frac{i}{d}(c-E_4)\end{pmatrix} \\ &= \left(e^{-iE_3t/\hbar}\frac{c-E_4}{E_3-E_4} + e^{-iE_4t/\hbar}\frac{E_3-c}{E_3-E_4}\right)|2\rangle \\ &\quad + \frac{i}{d}\frac{(c-E_4)(c-E_3)}{E_3-E_4}\left(e^{-iE_3t/\hbar} - e^{-iE_4t/\hbar}\right)|3\rangle. \end{aligned}$$

Problem 3: Displaced Harmonic Oscillator in Matrix Representation

(a) We have $H = p^2/2m + 1/2\, m\omega^2x^2 + m\omega^2 bx$, where I have substituted b for a to avoid confusion with the raising and lowering operators. Using the definitions of the raising and lowering operators in terms of $\hat{x}$ and $\hat{p}$, we find that

$$x = \sqrt{\frac{\hbar}{2m\omega}}\,(a+a^\dagger) \tag{4}$$

so that

$$\begin{aligned} H &= \hbar\omega\left(\hat{N}+\frac{1}{2}\right) + m\omega^2 b\sqrt{\frac{\hbar}{2m\omega}}(a+a^\dagger) \\ &= \hbar\omega\left(\hat{N}+\frac{1}{2}\right) + \hbar\omega\left[b\sqrt{\frac{m\omega}{2\hbar}}\,(a+a^\dagger)\right]. \end{aligned}$$

We are representing H in the basis of undisplaced oscillator states. The action of the raising and lowering operators on these states is known: $a\,|n\rangle = \sqrt{n}\,|n-1\rangle$ and $a^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle$. Also $\hat{N}\,|n\rangle = a^\dagger a\,|n\rangle = n\,|n\rangle$. We form the matrix representation of H by computing $\langle m|\,H\,|n\rangle$ for all m,n, and putting the result in the m^{th} row and n^{th} column of the matrix. Since the basis states are orthogonal, we have nonzero entries only along the main diagonal and along the two diagonals on either side of the main diagonal.

Making the definition $\lambda \equiv b\sqrt{\frac{m\omega}{2\hbar}}$, the matrix representation of H can thus be written as

$$\text{H}= \hbar\omega \begin{pmatrix} 1/2 & \lambda\sqrt{1} & 0 & 0 & 0 & \dots \\ \lambda\sqrt{1} & 3/2 & \lambda\sqrt{2} & 0 & 0 & \dots \\ 0 & \lambda\sqrt{2} & 5/2 & \lambda\sqrt{3} & 0 & \dots \\ 0 & 0 & \lambda\sqrt{3} & 7/2 & \lambda\sqrt{4} & \dots \\ 0 & 0 & 0 & \lambda\sqrt{4} & 9/2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{5}$$

(b) The potential $V(x)$ can be written as

$$\begin{aligned} V(x) &= \frac{1}{2}m\omega^2\left(x^2+2bx\right) \\ &= \frac{1}{2}m\omega^2(x+b)^2 - \hbar\omega\left(\frac{m\omega b^2}{2\hbar}\right) \ . \end{aligned}$$

Hence if we just redefine our coordinate axes to be centered at $-b$, then we would get the old harmonic oscillator wave functions again, except that the energies are shifted by the constant term that appears in the hamiltonian.

(c) Define c_n by

$$|0\rangle_a = \sum_{n=0}^{\infty} c_n \, |n\rangle_o \ . \tag{6}$$

Then the m^{th} row of the matrix equation

$$\begin{pmatrix} \\ H_{m0} \ H_{m1} \ H_{m2} \ \dots \\ \\ \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ \vdots \end{pmatrix} = E_{0a} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_m \\ \vdots \end{pmatrix} \tag{7}$$

which gives

$$\sum_{n=0}^{\infty} H_{mn} c_n = E_{oa} c_m \tag{8}$$

which is equivalent to the given expression.

(d) Multiply the first few rows of H give the following equations:

$$\begin{aligned} \frac{1}{2}c_o + \lambda c_1 &= \frac{E_{oa}}{\hbar\omega}c_o \\ \lambda c_o + \frac{3}{2}c_1 + \lambda\sqrt{2}c_2 &= \frac{E_{oa}}{\hbar\omega}c_1 \\ \lambda\sqrt{2}c_1 + \frac{5}{2}c_2 + \lambda\sqrt{3}c_3 &= \frac{E_{oa}}{\hbar\omega}c_2 \\ \lambda\sqrt{3}c_2 + \frac{7}{2}c_3 + \lambda\sqrt{4}c_4 &= \frac{E_{oa}}{\hbar\omega}c_3 \\ \lambda\sqrt{4}c_3 + \frac{9}{2}c_4 + \lambda\sqrt{5}c_5 &= \frac{E_{oa}}{\hbar\omega}c_5 \ \dots \end{aligned}$$

Using these equations, we can first solve for c_1 in terms of c_o, then use the next equation to write c_2 in terms of c_o, and so forth. The first equation gives $c_1 = -\lambda c_0$; substituting into the next equation gives $c_2 = (\lambda^2/\sqrt{2})c_o$; then $c_3 = (-\lambda^3/\sqrt{6})c_o$, and so forth. The pattern soon becomes clear:

$$c_n = \frac{(-\lambda)^n}{\sqrt{n!}} c_o \tag{9}$$

The normalization requirement will give us a condition on c_o:

$$\sum_{n=0}^{\infty} |c_n|^2 = |c_o|^2 \sum_{n=0}^{\infty} \frac{(-\lambda)^{2n}}{n!} = |c_o|^2 e^{\lambda^2} = 1 \ ;$$
$$|c_o|^2 = e^{-\lambda^2} \ ,$$
$$c_o = e^{-\lambda^2/2} \ .$$

Subsituting back in for λ we obtain finally

$$c_n = exp\left(-\frac{m\omega b^2}{4\hbar}\right)\left(-b\sqrt{\frac{m\omega}{2\hbar}}\right)^n \frac{1}{\sqrt{n!}} \ . \tag{10}$$