RECURSIVE EVALUATION OF 3j AND 6j COEFFICIENTS *

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PROGRAM SUMMARY

Title of program: J1-RECURSION OF 3J-COEFFICIENTS
Catalogue number: ACWQ

Program obtainable from: CPC Program Library, Queen's
University of Belfast, N. Ireland (see application form in this
issue)

Computer: UNIVAC 1108; Installation: Gesellschaft für
Germany.

Operating system: EXEC 8
Program language used: FORTRAN 4
High speed storage required: 8958 words

No. of bits in a word: 36
Overlay structure: None

No. of magnetic tapes required: None

Other peripherals used: Card reader, lineprinter (for test)
No. of cards in combined program and test deck: 405

Card punching code: BCD

Keywords: General purpose, molecular, rotation group, re-
coupling coefficient, 3j, Clebsch–Gordan, Wigner, angular
momentum, recursion.

Nature of physical problem
Subroutine RECSJJ generates 3j coefficients

\[ f(j_1) = \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \]

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for all allowed \( j_1 \) \((j_2, j_3, m_1, m_2, m_3 \) held fixed) from the
exact solution of a recursion equation. The algorithm is more
efficient and accurate than those based on explicit expres-
sions, particularly in the commonly arising case in which a
complete set of 3j coefficients is needed. The algorithm is
numerically stable for large quantum numbers which occur in
problems of molecular dynamics.

Methods of solution
To guarantee numerical stability the recursion equation which
relates 3j coefficients \( f(j_1) \) with contiguous \( j_1 \) values \( j_1 - 1, \)
\( j_1, j_1 + 1 \) is solved in the direction of increasing \( f(j_1) \) from
both ends of the allowed \( j_1 \) domain, \( j_1 \text{min} \) and \( j_1 \text{max} \). The
linear recursion equation reduces to two terms at \( j_1 \text{max} \) and
\( j_1 \text{min} \) and thus can be started at both ends with arbitrary
initial values \( f(j_1 \text{min}) \) and \( f(j_1 \text{max}) \), respectively. At an inter-
mediate \( j_1 \) forward and backward recursions are matched
which leaves all \( f(j_1) \) off by a constant factor. This factor is
determined from the unitary property of 3j coefficients and
Wigner's phase convention.

Typical running time
0.4 msec per 3j coefficient for \( j_1 \text{max} - j_1 \text{min} > 20 \), somewhat
longer for smaller \( j_1 \) domains.

Unusual features of the program
Large quantum number 3j coefficients \( f(j_1) \) may vary over
many orders of magnitude over their \( j_1 \) domain. The program
prevents underflow and overflow for which purpose the small-
est and largest number representable on the computer, TINY
and HUGE, respectively, have to be defined. In the recursion
process the relative magnitudes of contiguous 3j coefficients
\( f(j_1) \) are being evaluated exactly, however. The program sets
later on all 3j coefficients which are smaller than TINY to zero.
PROGRAM SUMMARY

Title of program: M2-RECURSION OF 3J-COEFFICIENTS
Catalogue number: ACWR
Program obtainable from: CPC Program Library, Queen's University of Belfast, N. Ireland (see application form in this issue)
Operating system: EXEC 8
Program language used: FORTRAN 4
High speed storage required: 8879 words
No. of bits in a word: 36
Overlay structure: None
No. of magnetic tapes required: None
Other peripherals used: Card reader, lineprinter (for test)
No. of cards in combined program and test deck: 392
Card punching code: BCD

Keywords: General purpose, molecular, rotation group, recoupling coefficient, 3j, Clebsch–Gordan, Wigner, angular momentum, recursion.

Nature of physical problem
Subroutine REC3JM generates 3j coefficients

\[ g(m_2) = \begin{pmatrix} i_1 & i_2 & i_3 \\ m_1 & m_2 & -m_1 - m_2 \end{pmatrix} \]

for all allowed \( m_2 (j_1, j_2, j_3, m_1 \) held fixed) from the exact solution of an recursion equation. The algorithm is more efficient and accurate than those based an explicit expressions, particularly, in the commonly arising case in which a complete set of 3j coefficients is needed. The algorithm is numerically stable for large quantum numbers which occur in problems of molecular dynamics.

Methods of solution
To guarantee numerical stability the recursion equation which relates 3j coefficients \( g(m_2) \) with contiguous \( m_2 \) values \( m_2 - 1, m_2, m_2 + 1 \) is solved in the direction of increasing \( g(m_2) \) from both ends of the allowed \( m_2 \) domain, \( m_2 \text{min} \) and \( m_2 \text{max} \). The linear recursion equation reduces to two terms at \( m_2 \text{min} \) and \( m_2 \text{max} \) and thus can be started at both ends with arbitrary initial values \( g(m_2 \text{min}) \) and \( g(m_2 \text{max}) \), respectively. At an intermediate \( m_2 \) forward and backward recursion are matched which leaves all \( g(m_2) \) off by a constant factor. This factor is determined from the unitary property of 3j coefficients and Wigner’s phase convention.

Typical running time
0.3 msec per 3j coefficient for \( m_2 \text{max} - m_2 \text{min} > 20 \), somewhat longer for smaller \( m_2 \) domains.

Unusual features of the program
Large quantum number 3j coefficients \( g(m_2) \) may vary over many orders of magnitude over their \( m_2 \) domain. The program prevents underflow and overflow for which purpose the smallest and largest number representable on the computer, TINY and HUGE, respectively, have to be defined. In the recursion process the relative magnitude of contiguous 3j coefficients \( g(m_2) \) are being evaluated exactly, however, the program sets later on all 3j coefficients which are smaller than TINY to zero.

PROGRAM SUMMARY

Title of program: J1-RECURSION OF 6J-COEFFICIENTS
Catalogue number: ACWS
Program obtainable from: CPC Program Library, Queen's University of Belfast, N. Ireland (see application form in this issue)
Operating system: EXEC 8
Program language used: FORTRAN 4
High speed storage required: 9178 words
No. of bits in a word: 36

Overlay structure: None
No. of magnetic tapes required: None
Other peripherals used: Card reader, lineprinter (for tests)
No. of cards in combined program and test deck: 431
Card punching code: BCD

Keywords: General purpose, molecular, rotation group, recoupling coefficient, 6j, Racah, Wigner, angular momentum, recursion.

Nature of physical problem
Subroutine REC6J generates 6j coefficients

\[ h(j_1) = \begin{pmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{pmatrix} \]
for all allowed \( j_1, j_2, j_3, l_1, l_2, l_3 \) held fixed) from the exact solution of a recursion equation. The algorithm is more efficient and accurate then those based on explicit expressions, particularly in the commonly arising case in which a complete set of \( 6j \) coefficients is needed. The algorithm is numerically stable for large quantum numbers which occur in problems of molecular dynamics.

Methods of solution

To guarantee numerical stability the recursion equation which relates \( 6j \) coefficients \( h(j_1) \) with contiguous \( j_1 \) values \( j_1 - 1, j_1, j_1 + 1 \) is solved in the direction of increasing \( j_1 \) from both ends of the allowed \( j_1 \) domain, \( j_1 \) min and \( j_1 \) max. The linear recursion equation reduces to two terms at \( j_1 \) min and \( j_1 \) max and thus can be started at both ends with arbitrary initial values \( h(j_1) \) min and \( h(j_1) \) max, respectively. At an intermediate \( j_1 \) forward and backward recursions are matched which leaves all \( f(j_1) \) off by a constant factor. This factor is determined from the unitary property of \( 6j \) coefficients and Wigner's phase convention.

Typical running time

0.5 msec per \( 6j \) coefficient for \( j_1 \) max > 20, somewhat longer for smaller \( j_1 \) domains.

Unusual features of the program

Large quantum number \( 6j \) coefficients \( h(j_1) \) may vary over many orders of magnitude over their \( j_1 \) domain. The program prevents underflow and overflow for which purpose the smallest and largest number representable on the computer, TINY and HUGE, respectively, have to be defined. In the recursion process the relative magnitudes of contiguous \( 6j \) coefficients \( h(j_1) \) are being evaluated exactly, however. The program sets later on all \( 6j \) coefficients which are smaller than TINY to zero.

LONG WRITE-UP

1. Introduction

\( 3j \) and \( 6j \) coefficients occur in the quantum mechanical algebra of angular momentum addition. Calculations involving the coupling of angular momenta commonly require the evaluation of whole strings of coupling coefficients of the kind

\[
f(j) = \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}, \quad \text{for all allowed } j_1; \quad j_1 \text{ min} \leq j_1 \leq j_1 \text{ max},
\]

(1a)

\[
g(m) = \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_1 - m_2 \end{pmatrix}, \quad \text{for all allowed } m_2; \quad m_2 \text{ min} \leq m_2 \leq m_2 \text{ max},
\]

(2a)

\[
h(j) = \begin{pmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{pmatrix}, \quad \text{for all allowed } j_1; \quad j_1 \text{ min} \leq j_1 \leq j_1 \text{ max}.
\]

(3a)

Existing algorithms evaluate coupling coefficients separately and do not make use of relationships between the values of contiguous \( 3j \) and \( 6j \) coefficients in eqs. (1a), (2a) and (3a). The algorithms, furthermore, are inapplicable for large angular momentum quantum numbers (~ 100) which, for example, frequently occur in problems of molecular dynamics.

We have pointed out recently that \( 3j \) and \( 6j \) coefficients can be evaluated most efficiently and accurately from recursion equations [1]. Condon and Shortley [2], and subsequently Rose [3] have called attention to this possibility for the evaluation of \( 3j \) coefficients. The recursion equations are particularly suitable for the evaluation of large quantum number coupling coefficients. In fact, they yield in the limit of very large quantum numbers a simple second-order difference equation from which the semiclassical expressions of \( 3j \) and \( 6j \) coefficients follow by means of the WKB approximation [4]. The semiclassical expressions reveal that the quantum number domain of \( 3j \) and \( 6j \) coefficients in eqs. (1a), (2a) and (3a) are divided in two non-classical domains separated by a classical domain. In the non-classical domains at the boundaries \( j_1 \) min and \( j_1 \) max (\( m_2 \) min and \( m_2 \) max) the coupling coeffi-
coefficients decay exponentially to zero. In the intermediate classical domain the values of the coupling coefficients
oscillate rapidly. This behaviour is somewhat reminiscent of the behaviour of bound state wave functions to the
Hamilton operator. Indeed, the recursion equations of $3j$ and $6j$ coefficients given below follow directly from
eigenvalue problems which define the coupling coefficients. They are also solved in a way similar to the integration
of bound state Schrödinger equations. The recursion equations for the coupling coefficients in eqs. (1a), (2a) and
(3a) are

$$j_1 A(j_1 + 1) \begin{pmatrix} j_1 + 1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} + B(j_1) \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} + (j_1 + 1) A(j_1) \begin{pmatrix} j_1 - 1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = 0,$$

where

$$A(j_1) = [(j_2 - j_3)^2]^{1/2} [(j_2 + j_3 + 1)^2 - j_1^2]^{1/2} [j_1^2 - m_1^2]^{1/2},$$

$$B(j_1) = -(2j_1 + 1) [j_2 (j_2 + 1) m_1 - j_3 (j_3 + 1) m_1 - j_1 (j_1 + 1) (m_1 - m_2)],$$

and

$$C(m_2 + 1) \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 + 1 & m_3 - 1 \end{pmatrix} + D(m_2) \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} + C(m_2) \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 - 1 & m_3 + 1 \end{pmatrix} = 0,$$

where

$$C(m_2) = [(j_2 - m_2 + 1) (j_2 + m_2) (j_3 + m_3 + 1) (j_3 - m_3)]^{1/2},$$

$$D(m_2) = j_1 (j_1 + 1) + j_3 (j_3 + 1) - j_1 (j_1 + 1) + 2 m_2 m_3,$$

and

$$j_1 E(j_1 + 1) \begin{pmatrix} j_1 + 1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{pmatrix} + F(j_1) \begin{pmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{pmatrix} + (j_1 + 1) E(j_1) \begin{pmatrix} j_1 - 1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{pmatrix} = 0,$$

where

$$E(j_1) = [(j_1^2 - (j_2 - j_3)^2)] [(j_2 + j_3 + 1)^2 - j_1^2] [l_1^2 - (l_2 - l_3)^2] [l_2 + l_3 + 1)^2 - j_1^2]^{1/2},$$

$$F(j_1) = (2j_1 + 1) [j_1 (j_1 + 1) ([j_1 (j_1 + 1)] + j_2 (j_2 + 1) + j_3 (j_3 + 1)] + l_2 (l_2 + 1) [l_1 (l_1 + 1) + j_2 (j_2 + 1) - j_3 (j_3 + 1)]$$

$$+ l_3 (l_3 + 1) [l_1 (l_1 + 1) - j_2 (j_2 + 1) + j_3 (j_3 + 1)] - 2j_1 (j_1 + 1) l_1 (l_1 + 1) \}.$$

These linear recursion equations determine the coupling coefficients except for an overall constant factor which
can be obtained from the following unitary properties and phase conventions:

$$\sum_{j_1} (2j_1 + 1) \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}^2 = 1,$$

$$\text{sign} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_2 - j_3 - m_1} \text{,}$$

and

$$\sum_{m_2} (2j_1 + 1) \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_1 - m_2 \end{pmatrix}^2 = 1,$$

$$\text{sign} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 \text{max} & -m_1 - m_2 \text{max} \end{pmatrix} = (-1)^{j_2 - j_3 - m_1} \text{,}$$

(1e, f)
and
\[ \sum_{j_1}(2j_1+1)(2j_1+1)\binom{j_1}{l_1}^2 = 1, \]
\[ \text{sign} \begin{pmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{pmatrix} = (-1)^{j_2+j_3+l_2+l_3}. \] (3e,f)

2. Method of solution

The linear three-term recursion equations (1b), (2b) and (3b) reduce to two terms at the boundaries \( j_{1\text{min}} \) and \( j_{1\text{max}} \) (\( m_{2\text{min}} \) and \( m_{2\text{max}} \)). Hence, the recursion process can be started with a single starting value \( F(j_{1\text{min}}) \) \( [G(m_{2\text{min}}), H(j_{1\text{min}})] \) for the forward direction or \( F(j_{1\text{max}})[G(m_{2\text{max}}), H(j_{1\text{max}})] \) for the backward direction. Due to the linear character of the recursion equations these starting values can be chosen arbitrarily leaving the generated recursion series \( \{F(j_1)\}, \{G(m_2)\} \) or \( \{H(j_1)\} \) multiplied by a constant factor.

\[ F(j_1) = c_1 f(j_1), \text{ for all } j_1; \quad G(m_2) = c_2 g(m_2), \text{ for all } m_2; \quad H(j_1) = c_3 h(j_1), \text{ for all } j_1. \] (1g, 2g, 3g)

The unknown constants \( c_1, c_2 \) or \( c_3 \) can be determined through
\[ c_1 = (-1)^{j_2-j_3-m_1} \text{ sign}[F(j_{1\text{max}})]/\left[ \sum_{j_1}(2j_1+1)F(j_1)^2 \right]^{1/2}, \] (1f)
\[ c_2 = (-1)^{j_2-j_3-m_1} \text{ sign}[G(m_{2\text{max}})]/\left[ \sum_{m_2}(2j_1+1)G(m_2)^2 \right]^{1/2}, \] (2f)
\[ c_3 = (-1)^{j_2+j_3+l_2+l_3} \text{ sign}[H(j_{1\text{max}})]/\left[ \sum_{j_1}(2j_1+1)(2j_1+1)H(j_1)^2 \right]^{1/2}. \] (3f)

The signs follow from Wigner's phase convention (1f), (2f) and (3f), the absolute magnitudes from the normalization conditions (1e), (2e) and (3e).

The recursion procedure is stable only in the direction of increasing coupling coefficients. From the semiclassical expressions one observes that large quantum number 3j and 6j coefficients increase exponentially in the non-classical regions at both ends of the recursion domain [4]. Hence, the recursion must proceed simultaneously forward and backward from the two non-classical domains towards the intermediate classical domain of large coupling coefficients.

Let us take in the following the recursion of the 3j coefficients in eq. (1a) as an example to illustrate the numerical procedure. What will be said applies equally to the 3j and 6j coefficients in (2a) and (3a), respectively. One needs to generate the forward and backward recursion series
\[ \hat{F}(j_{1\text{min}}), \hat{F}(j_{1\text{min}}+1), \ldots, \hat{F}(j_{1\text{int}}), \quad \hat{F}(j_{1\text{max}}), \hat{F}(j_{1\text{max}}-1), \ldots, \hat{F}(j_{1\text{int}}), \]
each started with arbitrary values. These series have to be matched at an intermediate \( j_{1\text{int}} \) value \( j_{1\text{int}} \). For this purpose one may rescale either the forward or the backward recursion series such that \( \hat{F}(j_{1\text{int}}) = F(j_{1\text{int}}) \). If one chooses to rescale the forward recursion series, the scaling factor is
\[ \lambda = \hat{F}(j_{1\text{int}})/\hat{F}(j_{1\text{int}}). \] (4')

This expression is not suitable numerically since \( \hat{F}(j_{1\text{int}}) \) and \( \hat{F}(j_{1\text{int}}) \) may be small, perhaps zero, and connected with large relative errors. It is advantageous to match forward and backward recursion at three contiguous \( j_{1} \) values simultaneously by a least squares fit. The scaling factor \( \lambda \) is then determined such that
\[ [\lambda \hat{F}(j_{1\text{int}}+1)-\hat{F}(j_{1\text{int}}+1)]^2 + [\lambda \hat{F}(j_{1\text{int}})-\hat{F}(j_{1\text{int}})]^2 + [\lambda \hat{F}(j_{1\text{int}}-1)-\hat{F}(j_{1\text{int}}-1)]^2 \]
takes on its minimum value, i.e.
\[
\lambda = \frac{\bar{F}(j_{1\text{int}} + 1) + \bar{F}(j_{1\text{int}} - 1)}{\bar{F}(j_{1\text{int}} + 1)^2 + \bar{F}(j_{1\text{int}} - 1)^2},
\]
(4)
where \(j_{1\text{int}}\) must be chosen to lie within the classical domain where no three contiguous coupling coefficients are small. Recursion equation (1b) [and similarly eqs. (2b) and (3b)] enter the recursion program in the form
\[
f(j_1 + 1) = \alpha(j_1)f(j_1) + \beta(j_1)f(j_1 - 1),
\]
(5)
where \(|\beta(j_1)| \approx 1\). The semiclassical theory of angular momentum coupling which is applicable for moderate and large quantum numbers \([4]\) reveals that \(|\alpha(j_1)|\) takes on its minimum value in the classical domain and varies monotonically in the non-classical domains. Hence, by monitoring the variation of \(|\alpha(j_1)|\) in the forward recursion step a suitable point for forward and backward recursion can be found.

Large quantum number \(3j\) and \(6j\) coefficients may vary over many orders of magnitude. To prevent 'overflow' in the recursion step and 'underflow' in the normalization step the recursion series \(\bar{F}(j_1)\) and \(\bar{F}(j_1)\) are rescaled such that the largest \(\bar{F}(j_1) - [\bar{F}(j_1) - \text{values does not exceed SHRUG =}[\text{HUGE}]^{1/2}\) where \text{HUGE} is the largest number representable on the computer.

Those \(\bar{F}(j_1)\) \((\bar{F}(j_1))\) which fall then below \(\text{TINY}\), the smallest value representable on the computer, are set to zero. Thus, the recursion program evaluates the relative magnitudes of contiguous coupling coefficients exactly but may set the smallest \(3j\) and \(6j\) coefficients to zero.

3. Program structure

The recursive evaluation of the coupling coefficients in (1a), (2a) and (3a) is carried out in subroutines REC3JJ, REC3JM and REC6J, respectively. The subroutines are driven by test routines which read in the test data and determine the partition of recursion domains in non-classical and classical domains. The recursion routines are documented by comment cards which together with what has been said above allow the program steps to be easily followed.

3.1. Input

(a) For the generation of \(3j\) coefficients
\[
f(L_1) = \begin{pmatrix} L_1 & L_2 & L_3 \\ -M_2 - M_3 & M_2 & M_3 \end{pmatrix}, \quad \text{as in eq. (1a)},
\]
Card 1ff. FORMAT (4F10.1) \(L_2, L_3, M_2, M_3\).

(b) For the generation of \(3j\) coefficients
\[
g(M_2) = \begin{pmatrix} L_1 & L_2 & L_3 \\ M_1 & M_2 & -M_1 - M_2 \end{pmatrix}, \quad \text{as in eq. (2a)},
\]
Card 1ff. FORMAT (4F10.1) \(L_1, L_2, L_3, M_1\).

(c) For the generation of \(6j\) coefficients
\[
h(L_1) = \begin{pmatrix} L_1 & L_2 & L_3 \\ L_4 & L_5 & L_6 \end{pmatrix}, \quad \text{as in eq. (3a)},
\]
Card 1ff. FORMAT (5F10.1) \(L_2, L_3, L_4, L_5, L_6\).

3.2. Output

The string of \(3j\) and \(6j\) coefficients
\[
\begin{pmatrix} 100 & 60 \\ -10 & 60 & -50 \\ 120 & 60 & 70 \\ -10 & M_2 & 10 - M_2 \end{pmatrix}
\]
\(40 \leq L_1 \leq 160\), \(10^{-10}\) and \(\text{HUGE} = 10^{10}\). The test driving programs give the classical regions \(47.1 < L_1 < 114.6\) for (1a), \(-21.0 < M_2 < 30.1\) for (2a), and \(131.7 < L_1 < 189.9\) for (3a). Outside the classical regions the exponential decay of the coupling coefficients can be observed, inside the classical region the values of the coupling co-
efficient oscillate rapidly exhibiting ten nodes for the examples chosen. The matching point of forward and backward recursions lie well within the classical domain: $L_1 = 69$ for (1a), $M_2 = 5$ for (2a), and $L_1 = 159$ for (3a). The recursion series have been rescaled twice (1a), six times (2a) and four times (3a). All coupling coefficients with absolute values below $TINY = 10^{-10}$ have been set to zero. We have also added results for $3j$ and $6j$ coefficients with small quantum numbers.

Acknowledgement

The authors would like to thank the Gesellschaft für wissenschaftliche Datenverarbeitung mbH Göttingen for excellent facilities and assistance.

References

### TABLE OF 3J-COEFFICIENTS (RESCALED 0 TIMES)

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### TIME NEEDED (IN MILLISECONDS)

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### TABLE OF 3J-COEFFICIENTS (RESCALED 0 TIMES)

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**TIME NEEDED (IN MILLISECONDS):**

40.0

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**TIME NEEDED (IN MILLISECONDS):**

6.0

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**TIME NEEDED (IN MILLISECONDS):**

9.0
## TEST RUN OUTPUT FROM J1-RECURSION OF 6J-COEFFICIENTS

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## TIME NEEDED (IN MILLISECONDS)

| E1.0 |

## TABLE OF 6J-COEFFICIENTS (RESCALED 0 TIMES) | Li 10.0 15.0 | Li-DOMAIN (Q. MECH.) 1.0 -- 15.0 | RECURSIONS MATCHED AT 15.0 |
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## TIME NEEDED (IN MILLISECONDS)

| E8.0 |

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## TIME NEEDED (IN MILLISECONDS)

| E14.0 |