Storing Sequences of Biased Patterns in Neural Networks with Stochastic Dynamics †

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Abstract: A network of spin-like neurons with asymmetric exchange interactions and stochastic spike response is proposed. The network can store and recall time sequences of regular and random biased patterns. The patterns can overlap. The performance of the suggested network is described by Monte Carlo simulation, in terms of a Fokker-Planck equation and, for a very large number N of neurons, in terms of a Liouville equation. We provide analytical expressions for the timing of the recall and analyze the scatter of the recall around the limit of precise recall $N \to \infty$.

1. Introduction

Presently many investigators [2–7] begin to focus on neural networks capable of temporal recall since it is generally realized that brain function cannot be separated from the temporal dimension. The natural computational mode of the brain involves a continuous, ever changing stream of afferent (e.g. sensory) and efferent (e.g. motor control) data. Technical applications for neural computers will likewise require computational characteristics in the time domain, for example in robotics where one needs to recognize temporal patterns or to generate motor output. In this contribution we demonstrate the possibility to store and recall temporal sequences of patterns of control signals, in particular, we investigate the accuracy of such recall. The latter investigation appears to be expedient for it must be realized that scatter of temporal patterns, when it is additive in the time-domain, gives rise to a diffusion-like spread in time which must be controlled if precision is to be attained other than through sensory feedback.

Little [8], Hopfield [9] and others [10, 11] have shown that neural networks with spin-like neuronal units and symmetric synapses can be used as content-addressable memories. For this purpose the synaptic connections (exchange interactions W_{ik}) are chosen such that the equilibrium states of the network coincide with states which represent stored *static* patterns. Recently [1], we proposed a neural network model which solves the problem of a content-addressable memory for *temporal* patterns. Previous attempts to construct such memories have relied on non-symmetric synaptic interactions with temporal features: the synapses transmitted the interaction with a time delay [3–6] or

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NATO ASI Series, Vol. F41
Neural Computers
Edited by R. Eckmiller and Ch. v. d. Malsburg

the synapses changed their interactive strength in time [2, 7]. Such synapses have been observed only in very few and very specialized neural systems and, therefore, may not serve for temporal storage in brains.

We have shown in [1] that spin-like networks can realize storage of temporal patterns with synaptic interactions which are neither time–dependent nor exhibit a delay. The networks proposed in [1] require only two new features: (1) synaptic interactions must be sufficiently non-symmetric, (2) noise must be present. Both features are considered to be natural attributes of biological networks. The asymmetric synapses in our model as in models with time–dependent or delay synapses provide directed projections between quasi–equilibrium states of the network and, thereby, define a sequential order among stored patterns. Noise triggers transitions between consecutive patterns. Because of this stochastic attribute transition times are not precise. However, if stored patterns are represented by large enough sets of neurons, fluctuations in the transition time decrease.

The following contribution extends the investigation in [1] in two respects: (1) we consider the case that the patterns involved in the recall overlap (random biased patterns); (2) we analyze the scatter of the recall around the deterministic, i.e. precise, recall dynamics attained in the limit of infinitely large networks.

2. Model Network

Our network is composed of N neurons described by dynamic variables $\{S_i\}_{i=1}^{N}$. Neuron i is either firing ($S_i = 1$) or quiet ($S_i = 0$). The variables are updated asynchronously according to a probabilistic rule which represents the action of noise in the system

$$S_i\left(t + \frac{\tau}{N}\right) = \begin{cases} 1, & \text{with probability } f_i(t) \\ 0, & \text{with probability } 1 - f_i(t) \end{cases} \qquad (1)$$

where $f_i(h_i) = (1 + \exp[-(h_i - U)/T])^{-1}$ is the probability that neuron i fires at time $t + \tau/N$ if it is excited by the molecular field $h_i = \sum_k W_{ik} S_k$. The asynchronous updates of neurons establish a characteristic time scale of τ/N for one spin flip where τ corresponds to one Monte Carlo Step (MCS). The parameters U and T are the threshold potential and the network temperature. The threshold potential U defines the scale for neural interaction. In case of $h_i \gg U$ the neuron fires with certainty, in the opposite case ($h_i \ll U$) the neuron is quiescent. The temperature T, a measure of the fluctuations of neural potentials weights the excitation h_i of neuron i. In neural networks with strong fluctuations ($T \gg 1$) the neuron fires with probability $\frac{1}{2}$ independently of its interaction with other neurons. In case of vanishing temperature T the spike probability f_i converges to the Heaviside step function $\Theta(h_i)$, i.e. $\Theta(h_i) = 1$ for $h_i \geq U$ and $\Theta(h_i) = 0$ for $h_i < U$. The theory presented below requires only a monotonic behaviour of the function f_i with asymptotic values $f_i(\infty) = 1$ and $f_i(-\infty) = 0$, i.e. does not depend on the analytical expression given here.

The neural network described by (1) would be of Hopfield type and store a set of patterns $\mathbf{S}^\nu = \{S_i\}_{i=1}^{N}$, $\nu = 1, \ldots, m$ if the exchange interaction W_{ik}^0 were chosen

$W_{ik}^0 = N^{-1}\sum_i \left(2S_i^\nu - 1\right)\left(2S_k^\nu - 1\right)$. Associative storage in a Hopfield network requires that the stored patterns consist of fifty percent of active neurons and that the values S_i^ν are statistically independent such that the orthogonality condition $\sum_i \left(2S_i^\nu - 1\right) S_i^\mu = N/2\ \delta_{\nu\mu}$ is satisfied. The patterns $\mathbf{S}^\nu$ which we intend to store in our network are biased, i.e. consist of only a small fraction n^ν of firing neurons with $S_i^\nu = 1$, without the requirement of statistical independence of the S_i^ν $(n^\nu = \sum_i S_i^\nu)$. This allows to store meaningful, i.e. non–random, patterns. Storage of patterns with low level of activity [see also 12] appears to be closer to the behaviour found in brains. The impossibility to store sequences of unbiased patterns [9] is caused by the existence of spurious stable states, i.e. mixtures of only few patterns. In our network which stores biased patterns, there exist no spurious states which disturb the proper recall of the sequence, an observation which is also reported by Amit et al [12].

Storage is achieved in our network by a choice of exchange interactions which differ from that of the Hopfield network. We construct the synaptic interaction between neurons in a hierarchical manner. The formation of neural assemblies representing the stored patterns is achieved by an excitatory symmetric interaction

$$W_{ik}^0 = \sum_{\nu=1}^{m} \epsilon^\nu S_i^\nu S_k^\nu \tag{2}$$

where $\epsilon^\nu = 1/n^\nu$ is a normalization constant. The interaction (2) connects all those neurons which fire simultaneously in at least one of the m patterns by an excitatory synapse. In contrast to Hopfield's learning rule the large number of background neurons are not connected through excitatory $(W_{ik} > 0)$ interactions. This seems to be plausible since two neurons which belong to the background and, therefore, never fire simultaneously should not considered to be correlated with respect to their activity and, hence, should not interact directly.

With choice (2) for the interaction all patterns $\{S_i^\nu\}$ which overlap sufficiently with the initial state of the network will be retrieved, i.e. are represented in the asymptotic state of the network. In order to select a finite number of patterns one needs to invoke a negative (inhibitory) additional contribution to the exchange interaction

$$W_{ik}^I = -\sum_{\nu=1}^{m} \sum_{\substack{\mu=1 \\ |\nu-\mu|>1}}^{m} \gamma \frac{m}{N} S_i^\nu S_k^\mu . \tag{3}$$

Here γ is the average strength of the mutual inhibition, e.g. $\gamma = 1.0$. The particular inhibitory term chosen here introduces a competition between the neurons of any stored pattern ν and all other neurons except those of the preceeding and subsequent patterns $\nu \pm 1$ during the retrieval dynamics such that only that pattern μ closest to the initial state is represented asymptotically. The inhibitory term (3) provides the network with the ability to make a decision for a single pattern, for example to associate asymptotically to a mixed input pattern $0.7\mathbf{S}^\nu + 0.3\mathbf{S}^\mu$ the stored pattern $\mathbf{S}^\nu$ and not the superposition $\mathbf{S}^\nu + \mathbf{S}^\mu$.

In order to retrieve the patterns $\mathbf{S}^\nu$ in the sequence $\nu = 1, 2, \ldots, m$ we add positive foreward and negative backward interactions W_{ik}^p between consecutive patterns in the sequence, i.e.

$$W_{ik}^p = \sum_{\nu=1}^{m} \left(\alpha^{\nu-1} \epsilon^{\nu-1} S_k^{\nu-1} - \beta^{\nu+1} \epsilon^{\nu+1} S_k^{\nu+1} \right) S_i^\nu . \tag{4}$$

If a network is in state $\mathbf{S}^\nu$ the obvious effect of the added excitatory and inhibitory contributions is to excite $\mathbf{S}^{\nu+1}$ and, when this state is sufficiently presented, to inhibit $\mathbf{S}^\nu$. This induces a consecutive retrieval of patterns $\mathbf{S}^\sigma, \mathbf{S}^{\sigma+1}, \ldots, \mathbf{S}^m$ when the network starts in a state exhibiting dominant overlap with pattern $\mathbf{S}^\sigma$.

The actual synaptic strength between two neurons i and k is defined as follows

$$W_{ik} = \begin{cases} W_{ik}^0, & \text{if } \; W_{ik}^0 \neq 0 \\ W_{ik}^p, & \text{if } \; W_{ik}^0 = 0 \wedge \sum\limits_{\nu=1}^{m} S_i^\nu \left(S_k^{\nu-1} + S_k^{\nu+1} \right) \neq 0 \\ W_{ik}^I, & \text{if } \; \sum\limits_{\nu=1}^{m} \sum\limits_{\substack{\mu=1 \\ |\mu-\nu| \leq 1}}^{m} S_i^\nu S_k^\mu = 0 \end{cases} \tag{5}$$

This expression describes a hierarchical construction of neural interactions: all neurons which represent the same pattern are connected by excitatory synapses; all neurons which do not fire simultaneously in one of the m patterns but belong to succeeding patterns receive positive foreward and negative backward projections; all neurons which belong to different, not consecutive patterns inhibit each other strongly. To realize consecutive recall of patterns $\mathbf{S}^\sigma, \mathbf{S}^{\sigma+1}, \ldots, \mathbf{S}^m$ we found parameter values $U = 0.35$ and $T = 0.1$ suitable.

In case of exactly orthogonal patterns, i.e. for $\sum_i \epsilon^\nu S_i^\nu S_i^\mu = \delta_{\nu\mu}, \; \forall \mu, \nu$, (5) can be expressed in closed form

$$W_{ik} = \sum_{\nu=1}^{m} \left(\epsilon^\nu S_k^\nu - \sum_{\substack{\mu=1 \\ |\nu-\mu|>1}}^{m} \gamma \frac{m}{N} S_k^\mu + \alpha^{\nu-1} \epsilon^{\nu-1} S_k^{\nu-1} - \beta^{\nu+1} \epsilon^{\nu+1} S_k^{\nu+1} \right) S_i^\nu . \tag{6}$$

3. Simulation of Temporal Recall for Overlaping Patterns

The result of a Monte–Carlo simulation of our model is shown in Fig. 1. The network considered has stored 10 random biased patterns with a mean activity level of 0.1, i.e. only 10 percent of the neurons in each pattern are active. The correlations $C_{\nu\mu}$ between two patterns ν and μ, defined as the number of neurons which fire in both patterns is

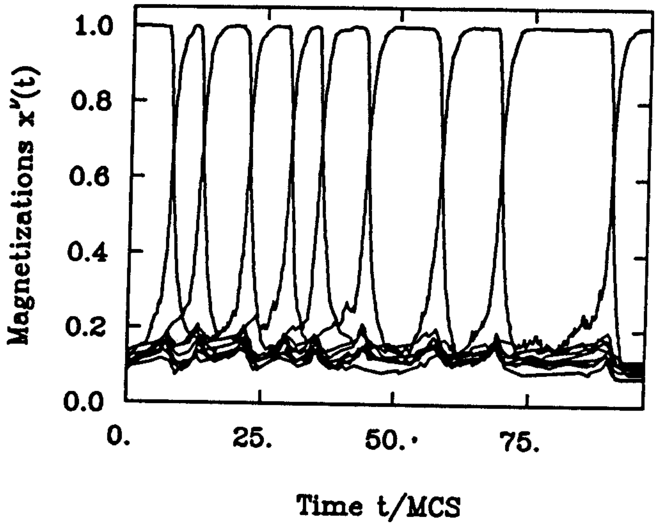

Fig. 1.
Simulation of the network which stores 10 random biased patterns with only 10 percent of the neurons active. The curves show the evolution of the magnetizations $x^\nu(t)$ as a function of time. The network consists of 3000 neurons which are all connected with each other. (Network parameters: $\alpha^\nu = -0.15, \beta^\nu = 2.0\,\forall\nu$, $\gamma = 2.0, T = 0.075, U = 0.35$)

given by the matrix $C_{\nu\mu} = \sum_i S_i^\nu S_i^\mu$. For the example studied $C_{\nu\mu}$ has the form

$$C_{\nu\mu} = \begin{pmatrix} 310 & 31 & 29 & 27 & 29 & 23 & 30 & 29 & 38 & 35 \\ 31 & 293 & 38 & 28 & 39 & 21 & 21 & 37 & 21 & 20 \\ 29 & 38 & 296 & 22 & 38 & 28 & 42 & 29 & 36 & 26 \\ 27 & 28 & 22 & 271 & 32 & 24 & 28 & 28 & 26 & 20 \\ 29 & 39 & 38 & 32 & 312 & 37 & 32 & 33 & 31 & 20 \\ 23 & 21 & 28 & 24 & 37 & 273 & 30 & 27 & 29 & 29 \\ 30 & 21 & 42 & 28 & 32 & 30 & 301 & 27 & 31 & 31 \\ 29 & 37 & 29 & 28 & 33 & 27 & 27 & 299 & 28 & 29 \\ 38 & 21 & 36 & 26 & 31 & 29 & 31 & 28 & 311 & 24 \\ 35 & 20 & 26 & 20 & 20 & 29 & 31 & 29 & 24 & 265 \end{pmatrix},$$

i.e. the patterns all overlap in about 1 percent of the neurons.

The *magnetizations* $x^\nu(t) = \sum_k \epsilon^\nu S_k^\nu S_k(t)$ which measure the overlap of pattern S^ν with the momentaneous network state $S(t)$ each are seen in Fig. 1 to assume small resting values around 0.1 except for a brief period when the x^ν rise close to the value one. The ordering of these periods implies that the network consecutively jumps from one pattern to the next.

4. Theory of Temporal Recall for Orthogonal Patterns

In the following we focus on the special, analytically tractable case of a network with patterns stored for which holds the orthogonality condition $\sum_i \epsilon^\nu S_i^\nu S_i^\mu = \delta_{\nu\mu}$ for all patterns ν and μ. In this case a salient feature in the choice of the exchange interaction (6) is that the molecular field h_i depends solely on the *magnetizations* $x^\nu(t) = \sum_i \epsilon^\nu S_i^\nu S_i(t)$ and not on the neuron index i. The network, therefore, can be completely described by the dynamic variables $x^\nu(t)$, $\nu = 1, \ldots, m$; the probability f_i in (1) can be replaced by

$$f^\nu\left(\mathrm{x}(t)\right) = \left(1 + exp\left(-\frac{\Sigma^\nu - U}{T}\right)\right)^{-1} \tag{7}$$
$$\Sigma^\nu = \alpha^{\nu-1}x^{\nu-1} + x^\nu - \beta^{\nu+1}x^{\nu+1} - \sum_{\substack{\mu=1 \\ |\nu-\mu|>1}}^{m} \gamma\frac{mn^\nu}{N}x^\mu .$$

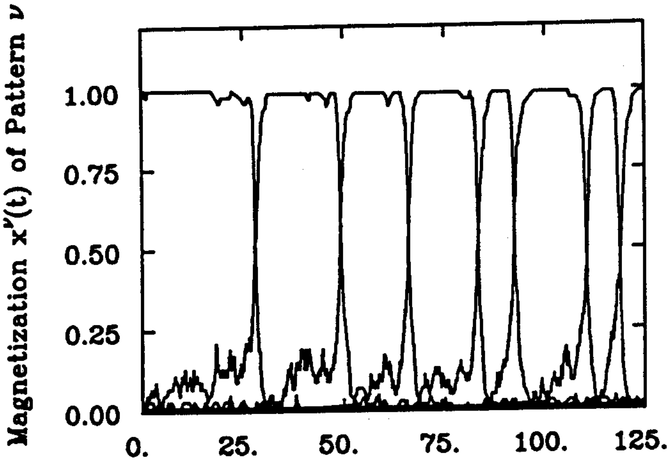

Fig. 2.
Simulation of the network with 8 patterns stored and parameters $\nu = 8, n^\nu = 100\ \forall\nu, \alpha^\nu = 0.1, \beta^\nu = 1.0, T = 0.1, U = 0.35$. The curves show the evolution of the magnetizations $x^\nu(t)$ as a function of time.

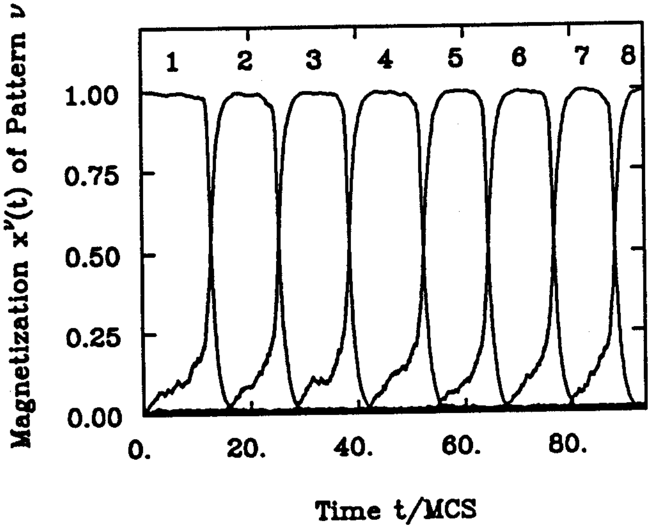

Fig. 3.
Simulation of the network with
$\nu = 8, n^\nu = 1000\ \forall\nu$,
$\alpha^\nu = 0.1, \beta^\nu = 1.0\ \forall\nu$,
$T = 0.1, U = 0.35$.

where Σ^ν is the molecular field of all neurons i for which hold $S_i^\nu = 1$. (Neurons which are not active in any stored pattern according to (6) are strongly inhibited by a molecular field $\Sigma = -\sum_\mu \gamma m n^\nu x^\mu / N$.) The introduction of the variables $x^\nu(t)$ drastically reduces the dimension of the phase space of the network, namely from 2^N to m. This reduction results from the fact that the parameters $\alpha^\nu, \beta^\nu, \gamma, U, T$ in (1,7) do not depend on the neuron index i. In case of networks with more heterogeneous interaction parameters, i.e. a neuron–dependent foreward projection strength α_i from pattern ν to $\nu + 1$, our model can also be applied when the interaction parameters are represented by their respective mean values.

The accuracy of the timing of the recall improves when the number of neurons which represent the patterns increases. To demonstrate this important aspect we present in Figs. 2,3 two Monte–Carlo simulations of a network model in which the patterns stored have the size $n^\nu = 100$ and $n^\nu = 1000$, respectively. The dynamics of the network considered is described by $x^\nu(t), \nu = 1, \ldots, 8$. A comparision of the results in Figs.2,3 demonstrates that fluctuations in the transition time decrease with the size of stored patterns.

To study the dynamics of our network we derive a master equation which describes the probability that the network assumes specific x^ν values at time t. A closed and simple master equation in terms of x^ν can be derived since the evolution of the network depends only on the magnetizations $\mathbf{x} = \{x^\nu(t)\}_{\nu=1}^m$ and since the asynchronous updates (1) affect always only a single $x^\nu(t)$.

4.1 Markov Process

Because of the asynchronous update of neurons different magnetizations x^ν are coupled only through the spike probability f^ν given in Eq. (7). (Note that x^νcan assume the values $0, \epsilon^\nu, 2\epsilon^\nu, \ldots, 1$.) Summing up all neural processes which result in the probability $p(\mathbf{x}, t)$ to find the system in a state with magnetization x^ν one obtains the rate equation

$$\begin{aligned} p\left(\mathbf{x}, t + \frac{\tau}{N}\right) = \sum_{\nu=1}^{m} \frac{1}{\epsilon^\nu N} \Big[& p\left(x^\nu + \epsilon^\nu\right)\left(x^\nu + \epsilon^\nu\right)\left(1 - f^\nu\left(x^\nu + \epsilon^\nu\right)\right) \\ + & \, p\left(x^\nu - \epsilon^\nu\right)\left(1 - x^\nu + \epsilon^\nu\right) f^\nu\left(x^\nu - \epsilon^\nu\right) \\ + & \, p\left(x^\nu\right) x^\nu f^\nu\left(x^\nu\right) + p\left(x^\nu\right)\left(1 - x^\nu\right)\left(1 - f^\nu\left(x^\nu\right)\right) \Big] \end{aligned} \tag{8}$$

where the arguments $x^\mu, \mu \neq \nu$ of $p(\mathbf{x}, t)$ and of $f^\nu(\mathbf{x}, t)$ which remain unaltered are suppressed on the r.h.s. The first term on the r.h.s. describes the transition of a firing neuron belonging to pattern ν to the resting state ($S_i(t) = 0$), the second term describes the inverse process. The last two terms account for processes which do not change the magnetization x^ν, i.e. that an active neuron fires again and a quiescent neuron remains in the resting state.

4.2 Fokker–Planck Equation

In the limit of many neurons ($N \to \infty$) and of large patterns ($\epsilon^\nu \to 0$) the discrete rate equation (8) becomes a continuous equation in time t and in the magnetizations $\mathbf{x}$. Taylor expansion of (8) results in the partial differential equation

$$\frac{\tau}{N} \frac{\partial}{\partial t} p(\mathbf{x}, t) = \sum_{\nu=1}^{m} \frac{n^\nu}{N} \left[- \sinh\left(\epsilon^\nu \frac{\partial}{\partial x^\nu}\right) F^\nu p + \left(\cosh\left(\epsilon^\nu \frac{\partial}{\partial x^\nu}\right) - 1\right) D^\nu p \right]$$

with $F^\nu = -(x^\nu - f^\nu)$ and $D^\nu = x^\nu + f^\nu - 2x^\nu f^\nu$. Neglecting terms of order $(\epsilon^\nu)^3$ and higher one obtains the Fokker–Planck equation

$$\tau \frac{\partial}{\partial t} p(\mathbf{x}, t) = \sum_{\nu=1}^{m} \left[- \frac{\partial}{\partial x^\nu} F^\nu p + \frac{\epsilon^\nu}{2} \frac{\partial^2}{\partial x^{\nu\,2}} D^\nu p \right] . \tag{9}$$

In the derivation of (9) we have assumed self-excitation of neurons ($W_{ii} \neq 0$). Without self-excitation the transition from the discrete rate equation to an equation continuous

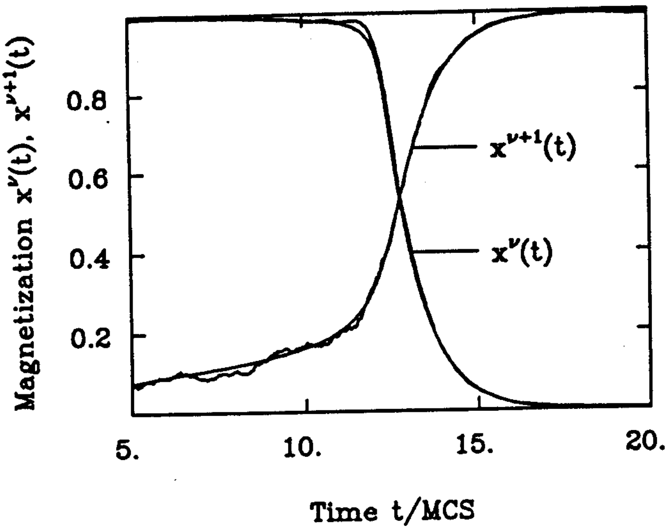

Fig. 4.
Time dependence of magnetizations $x^\nu(t)$ and $x^{\nu+1}(t)$. The solution of (10) is compared with the result from Monte Carlo simulations.

in x reproduces (9) except for an additional (diffusive) term $\sum_{\nu=1}^{m} 2\epsilon^\nu \left[\frac{\partial}{\partial x^\nu} x^\nu p \frac{\partial}{\partial x^\nu} f^\nu\right]$ on the r.h.s.

4.3 Liouville Equation

In the limit of infinitely large patterns ($\epsilon^\nu = 0$) Eq. (9) simplifies further and assumes the form of a deterministic Liouville equation [13]. In this limit the vector of magnetizations x obeys the kinetic equation

$$\tau \frac{d}{dt} x^\nu = -\left(x^\nu - f^\nu(\mathrm{x})\right). \tag{10}$$

An equation of this kind has also been derived by Peretto & Niez [14].

4.4 Estimate of Recall Parameters

The threshold U in (7) has to prevent fluctuations from activating patterns in wrong succession, but should not be as large as to suppress all pattern states during a transition from pattern ν to $\nu+1$ when x^ν and $x^{\nu+1}$ assume values around 0.5. To satisfy these requirements, U needs to lie in the range $T < U < 0.5$. Inserting such U into (7) yields $f^\nu \approx 1$ and $f^\mu \approx 0$, $\mu \notin \{\nu, \nu+1\}$ when the network occupies the ν^{th} pattern state, i.e. for $x^\mu = \delta_{\mu\nu}$. These considerations allow us to investigate the transition from pattern ν to $\nu+1$ by projecting the dynamics onto the plane spanned by $\{x^\nu, x^{\nu+1}\}$. In Fig. 4 the magnetizations $x^\nu(t)$ and $x^{\nu+1}(t)$ corresponding to the solution of (10) for the initial state $\left(x^\nu(0), x^{\nu+1}(0)\right) = (1,0)$ are compared with the magnetizations obtained from a Monte Carlo simulation.

We introduce now an analytical approximation for the transition of $(x^\nu, x^{\nu+1})$ from $(1,0)$ to $(0,1)$. This approximation allows us to determine some important network constants, e.g. the time for the transition from pattern ν to $\nu+1$ or the minimum value of the synaptic strength parameter α^ν needed for the suggested mechanism of temporal recall.

We first derive an estimate of the minimum α^ν value. For this purpose we consider the component $F^{\nu+1}$ along the axis $x^\nu = 1$. The minimum value $\Phi^{\nu+1}$ of $F^{\nu+1}\big|_{x^\nu=1}$ is

$$\begin{aligned} \Phi^{\nu+1} &= -\chi^{\nu+1} + \frac{2T}{1+\sqrt{1-4T}} \\ \chi^{\nu+1} &= U - \alpha^\nu - 2T\text{arcosh}\left(\frac{1}{2\sqrt{T}}\right) \end{aligned} \tag{11}$$

The condition $\Phi^{\nu+1} > 0$, which assures that the transition can be completed, determines the minimum value of α^ν.

To estimate the time needed for the transition we approximate the initial phase of the transition by means of a quadratic expansion of $F^{\nu+1}\big|_{x^\nu=1}$ around $\chi^{\nu+1}$ and by the corresponding solution of (10). The resulting magnetization $x^{\nu+1}$ coincides with the exact $x^{\nu+1}$ up to the time τ^* where assumption $x^\nu = 1$, equivalent to $f^\nu \approx 1$, fails. The time τ^*

$$\begin{aligned} \tau^* = \frac{\tau}{\sqrt{\Phi^{\nu+1}\omega}} \Bigg[& \arctan\left(\sqrt{\frac{\omega}{\Phi^{\nu+1}}}\chi^{\nu+1}\right) \\ & + \arctan\left(\sqrt{\frac{\omega}{\Phi^{\nu+1}}}\left(\frac{1-U_T^\nu-2T}{\beta^{\nu+1}} - \chi^{\nu+1}\right)\right)\Bigg]. \end{aligned} \tag{12}$$

has been obtained from the condition $1 - U - \beta^{\nu+1}x^{\nu+1} = 2T$, $(\omega = \sqrt{1-4T}/2T)$.

The behaviour of $x^\nu, x^{\nu+1}$ at times $t > \tau^*$ is dominated by the asymptotic values of f, i.e. $f^\nu \approx 0$ and $f^{\nu+1} \approx 1$. The magnetizations evolve asymptotically as $x_a^\nu(t) = \exp\left(-(t-\tau^\nu)/\tau\right)$, $x_a^{\nu+1}(t) = 1 - \exp\left(-(t-\tau^{\nu+1})/\tau\right)$ towards $(0,1)$, i.e. the $(\nu+1)^{th}$ pattern. The time constants τ^ν and $\tau^{\nu+1}$ involved can be determined numerically or approximated by $\tau^\nu \approx \tau^{\nu+1} \approx \tau^*$. The transition time $\tau_{\nu\to\nu+1}$ from pattern ν to $\nu+1$ is defined as the time spend between the moment when $x^{\nu+1}$ starts to grow, i.e. at $t=0$, and when, subsequently, $x^{\nu+2}$ starts to grow, i.e. when the minimal force $F_{min}^{\nu+2}$ becomes positive,

$$F_{min}^{\nu+2}(t) = -U + \alpha^{\nu+1}x_a^{\nu+1} - \gamma\frac{mn^\nu}{N}x_a^\nu + 2T\text{arcosh}\left(\frac{1}{2\sqrt{T}}\right) + \frac{2T}{1+\sqrt{1-4T}} > 0. \tag{13}$$

Solving (13) for time t we obtain the transition time $\tau_{\nu\to\nu+1}$ which consists of the terms τ^* and of the relaxation time into pattern $\nu+1$, i.e.

$$\tau_{\nu\to\nu+1} \approx \tau^* + \tau\ln\left(\frac{\alpha^{\nu+1}+\gamma mn^\nu/N}{\Phi^{\nu+2}}\right). \tag{14}$$

Figure 5 shows the dependence of the transition time $\tau_{\nu\to\nu+1}$ on the noise as measured by T for three different projection strengths α^ν. The transition time diverges for a critical temperature T^* defined by the condition $\Phi^{\nu+1} = 0$. If the minimal force which

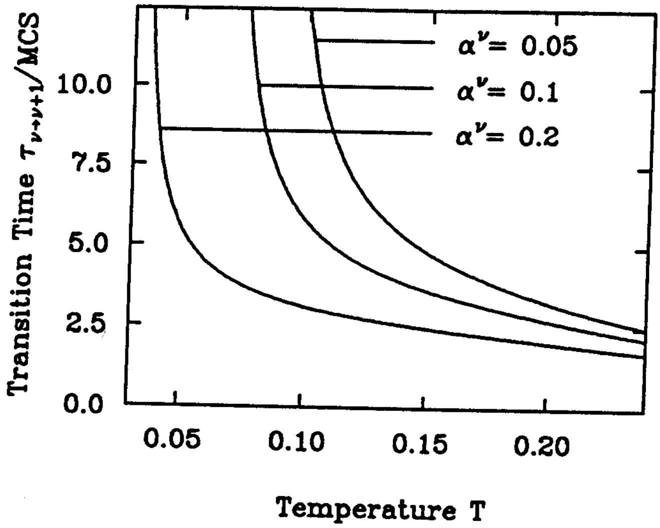

Fig. 5.
Transition time $\tau_{\nu\to\nu+1}$ as a function of temperature T
($\alpha^\nu = 0.1, \beta^\nu = 1.0, U = 0.35$).

drives the network towards the succeeding pattern $\nu+1$ vanishes the pattern state ν remains stable for an infinite time. In the limit $\epsilon^\nu \to 0$ one can actually tune the network to stay for an arbitrary time in the pattern state ν and then to jump to the next pattern $\nu+1$. Figure 5 also shows an alternative operation mode for the network proposed. By changing the temperature T in time, i.e. to switch from a temperature value below T^* to a value above T^*, the stability of the pattern states can be controlled. If T is smaller than T^* the network relaxes to the pattern state which has the largest overlap with the initial state and remains there for an arbitrary long period. For T larger than T^* the network evolves from one pattern state to the next according to the synaptic forward projection. In our network with time-dependent global noise the rythm of the pattern sequence reflects the variation of temperature T.

4.5 Accuracy of Recall

The analysis above for infinite networks can serve as an approximation for finite networks [13]. In the limit of small diffusion coefficients D^ν in (9) the dynamics of the magnetizations is dominated by the drift term. Fluctuations described by the diffusion term only induce a small broadening of the probability $p(\mathrm{x}(t))$. Van Kampen introduced a finite size approximation which can be applied in the limit of large but finite patterns. For this purpose (9) is transformed to variables

$$y^\nu = \frac{x^\nu - x^\nu(t)}{\sqrt{\epsilon^\nu}} \tag{15}$$

which measure the deviation from the solution $x^\nu(t)$ of (10). The resulting probability distribution $\Pi(\mathrm{y},t)$ for the y^ν obeys the m-dimensional linear Fokker–Planck equation

$$\tau \frac{\partial}{\partial t}\Pi(\mathrm{y},t) = \sum_{\nu=1}^{m}\left[-\sum_{\mu=1}^{m} \left.\frac{\partial F^\nu}{\partial(\sqrt{\epsilon^\mu}y^\mu)}\right|_{\mathrm{y}=0}\sqrt{\frac{\epsilon^\mu}{\epsilon^\nu}}\frac{\partial}{\partial y^\nu}y^\mu\Pi + \frac{D^\nu(\mathrm{x}(t))}{2}\frac{\partial^2}{\partial y^{\nu 2}}\Pi\right]. \tag{16}$$

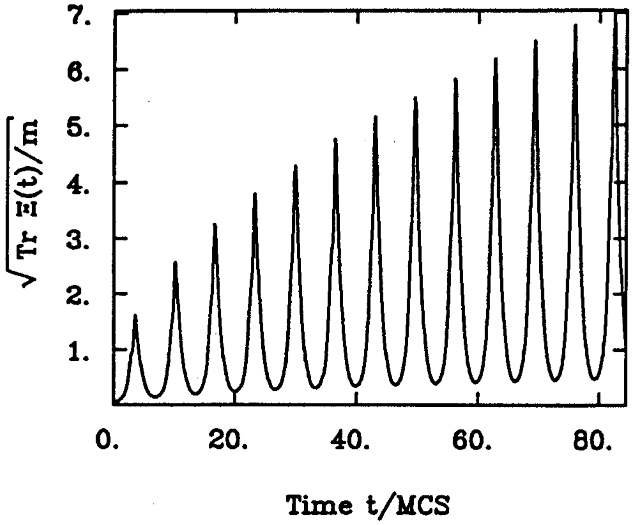

Fig. 6.
Radius of the distribution $\Pi(\mathbf{y},t)$, $\sqrt{Tr\Xi/m}$, as a function of time t. The network has stored 13 orthogonal patterns of equal size ($m = 13$). The minima of $\sqrt{Tr\Xi/m}$ evolves proportional to $\sqrt{t}$ as known for diffusive processes.

Equation (16) can be solved exactly and yields the m–dimensional Gaussian distribution

$$\Pi(\mathbf{y},t) \;=\; \frac{1}{\sqrt{(2\pi)^m Det\Xi}}\exp\left(-\frac{1}{2}\mathbf{y}\Xi^{-1}\mathbf{y}\right) \tag{17}$$

as shown by van Kampen [13]. The covariance matrix Ξ obeys the equation

$$\frac{d}{dt}\Xi \;=\; \mathbf{A}\Xi + \Xi\mathbf{A}^T + \mathbf{D}. \tag{18}$$

with the abbreviations

$$\mathbf{A}_{\nu\mu} \;=\; \left.\frac{\partial F^\nu}{\partial(\sqrt{\epsilon^\mu}y^\mu)}\right|_{\mathbf{y}=0}\sqrt{\frac{\epsilon^\mu}{\epsilon^\nu}}, \tag{19}$$

$$\mathbf{D} \;=\; \mathrm{diag}\left(D^1(\mathbf{x}(t)),\ldots,D^m(\mathbf{x}(t))\right) \tag{20}$$

If we choose a well–defined initial state $\mathbf{x}(0)$ all elements of $\Xi(t)$ vanish and $\Pi(\mathbf{y},0)$ assumes the form of a δ–function, i.e. $\Pi(\mathbf{y},0) = \delta(\mathbf{y})$.

Figure 6 shows the evolution of the radius $\sqrt{Tr\Xi/m}$ of the distribution $\Pi(\mathbf{y},t)$. During the transition from one pattern to the next $\Pi(\mathbf{y},t)$ is distorted by the force field F^ν and narrows when the network relaxes to the next pattern state. The peaks as well as the minima of the radius $\sqrt{Tr\Xi/m}$ evolve as $\sqrt{t}$ as expected for diffusive processes. It is obvious if the y^ν are inserted in (17) that the variance of the transition time of a single pattern transition decreases as $\sqrt{\epsilon^\nu}$ with increasing pattern size n^ν.

Another consideration concerns the question in how far patterns may overlap and still be properly recalled. An overlap is tolerable as long as it conserves the recall order of the patterns. However, such overlap can strongly affect the transition time between consecutive patterns as the random overlaps in the simulation of Fig. 1 demonstrates. The dynamics of a network with non–orthogonal patterns stored can be described in terms of the enlarged set of variables $x^{\nu_1,\ldots,\nu_k}(t)$ which count all neurons active at time t and belonging to all of the patterns $\nu_1,\ldots,\nu_k$. An analysis of the resulting dynamics shows that an overlap between consecutive patterns corresponds to an enhanced projection strength from ν to $\nu+1$. An appropriate choice [15] of α^{ν} allows to tune the transition time $\tau_{\nu\to\nu+1}$ for a given overlap $x^{\nu,\nu+1}$.

5. Summary

We have demonstrated that networks of spin–like neurons can store and recall time sequences of patterns by means of non-symmetric, time-independent and instantaneous exchange interactions. The result of a Monte Carlo simulation of a neural network which has stored random biased patterns demonstrates that also nonorthogonal patterns can be stored in our network. The transition between patterns during recall is triggered by global noise. The fluctuations of the transition times decrease with increasing network size. The scatter of pattern recall is investigated in terms of a finite size approximation for small diffusion coefficient and is found to obey a $\sqrt{t}$ law as typical for diffusive processes.

Acknowledgement: This work has been supported by the Deutsche Forschungsgemeinschaft (Schu 523/1-1).

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